Tomographic reconstruction techniques

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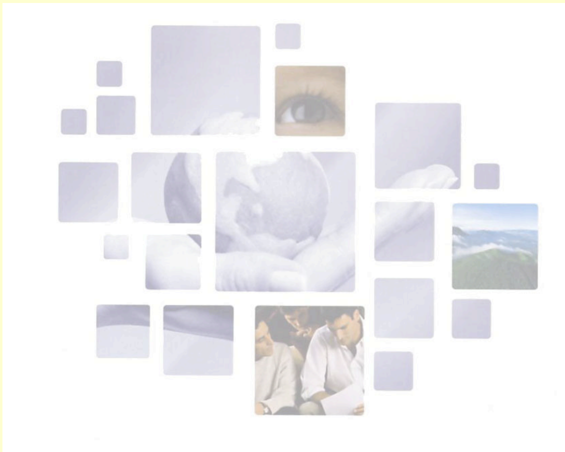

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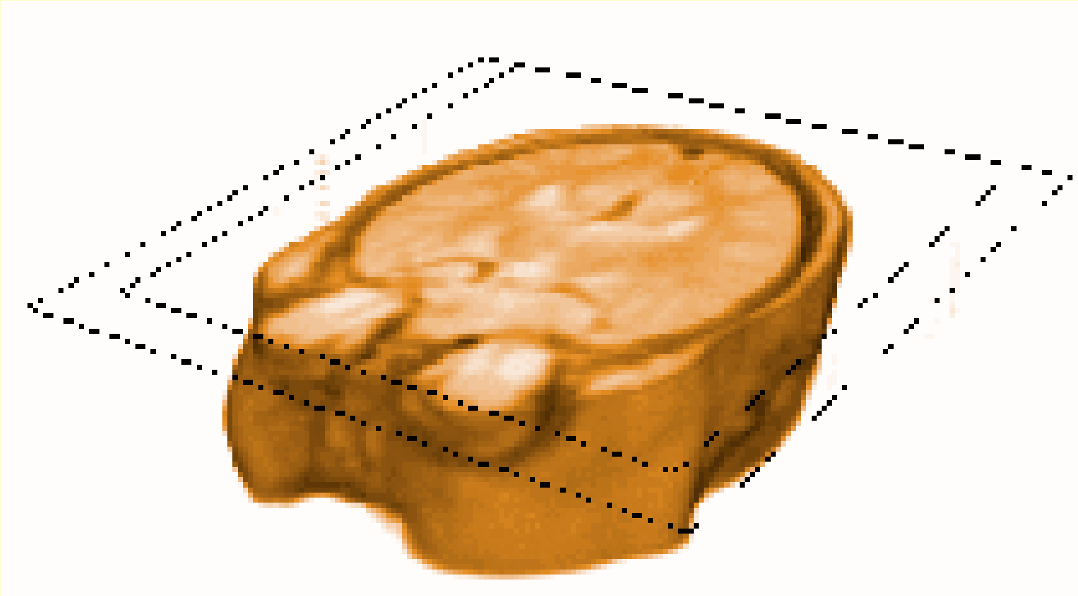
Outline

- Introduction: the issue of tomographic reconstruction
- Basics
- Tomographic reconstruction methods
- Regularization
- New challenges
- Conclusion

Introduction



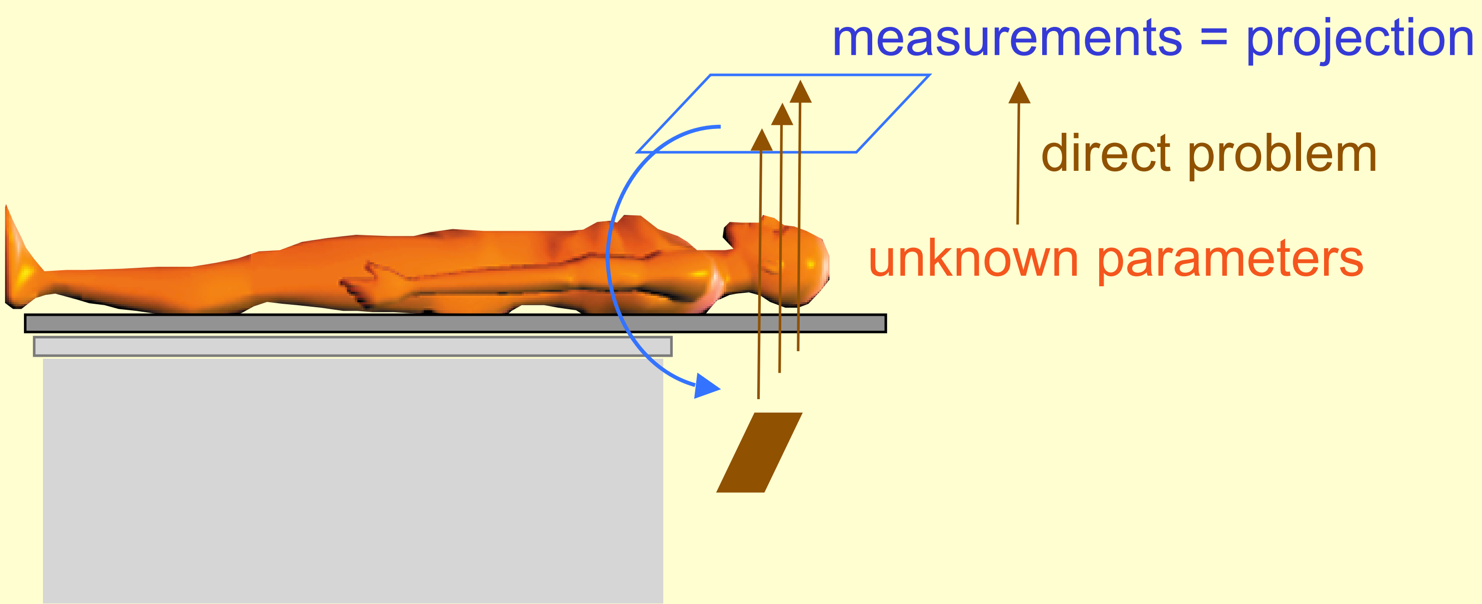
What is tomography ?

❶ An indirect measurement of a parameter of interest, using a detector sensitive to some sort of radiations

❷ Algorithms to recover the 3D cartography of the parameters from the measurements

Direct (forward) problem

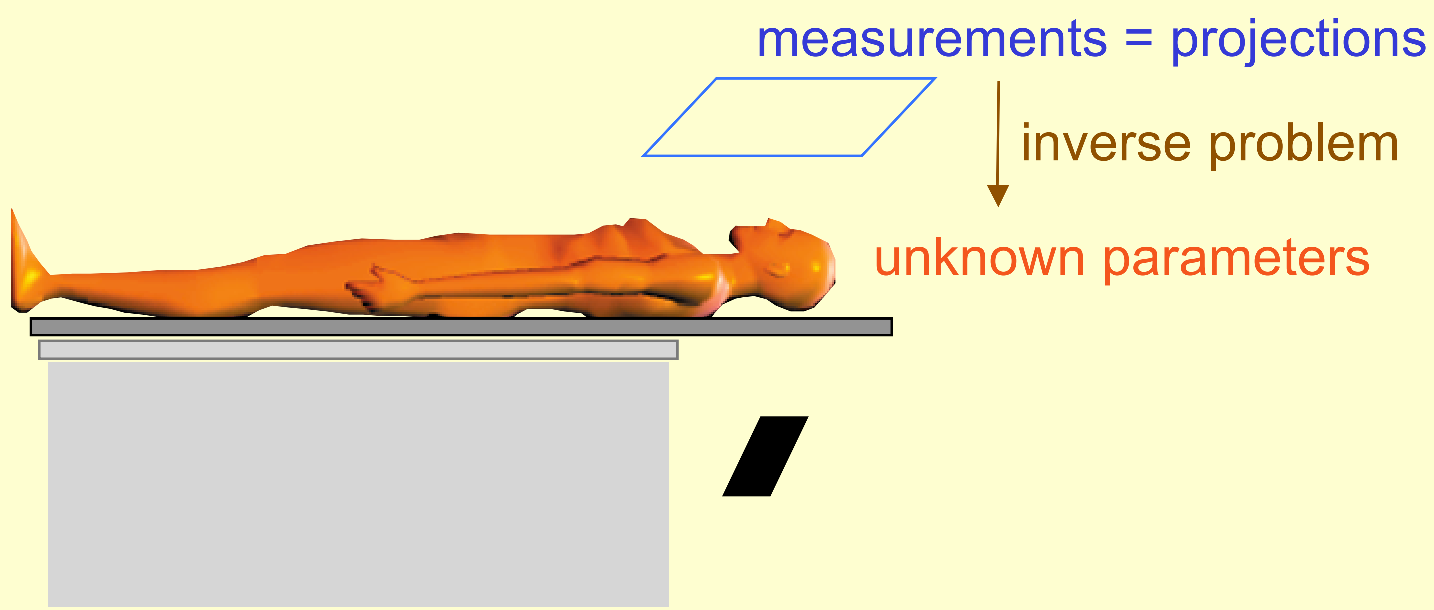
The tomographic system measures a set of “projection” data: integrals of a signal along certain specific directions.



The mathematical formulation of the relationship between the unknown parameters and the measurements is the direct problem.

Inverse problem

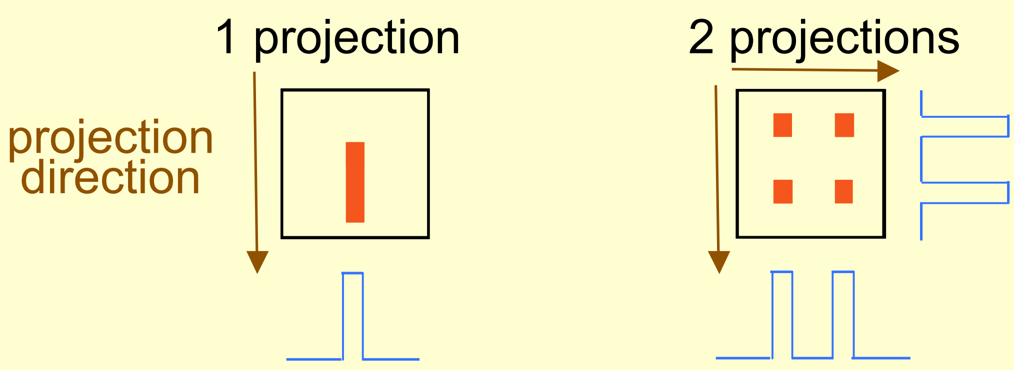
Tomographic reconstruction is the inversion of the direct problem.



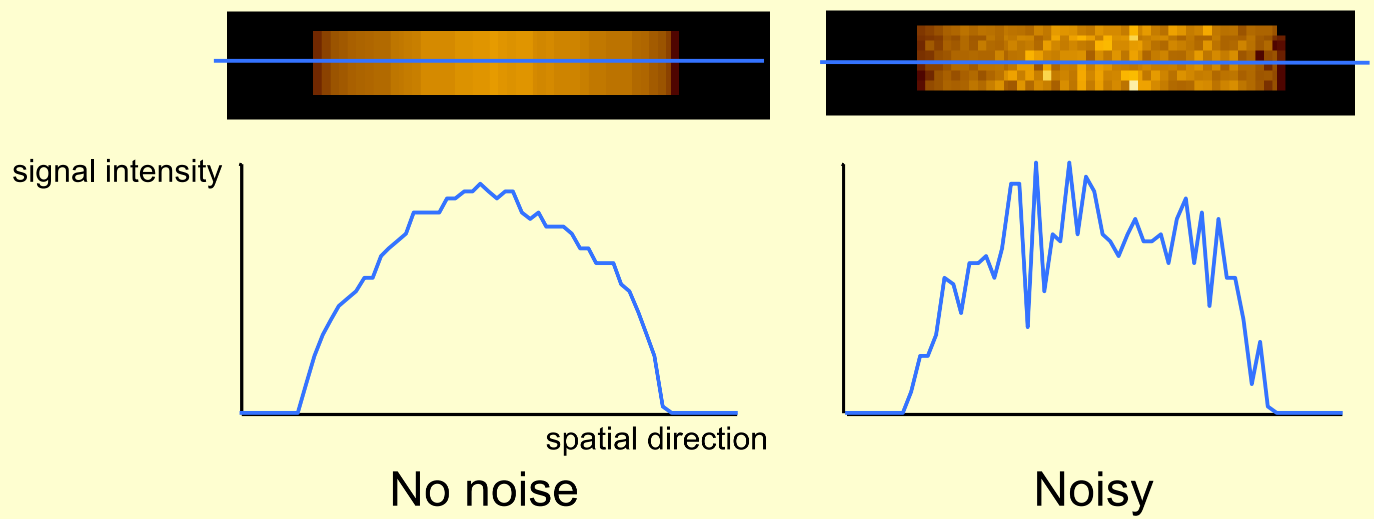
Inverse problem: estimating the 3D cartography of unknown parameters from the measured data.

Ill-posed inverse problem

Limited spatial sampling



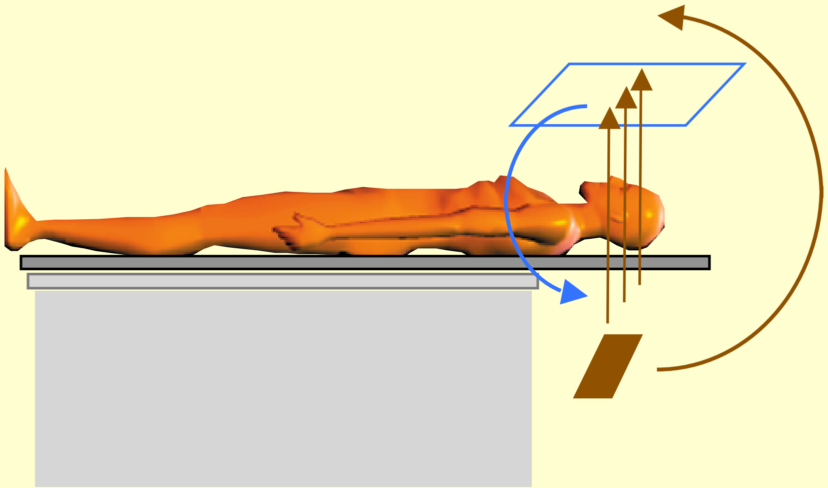
Noisy measurements



Ill-posed: several solutions compatible with the measurements

Different types of tomography

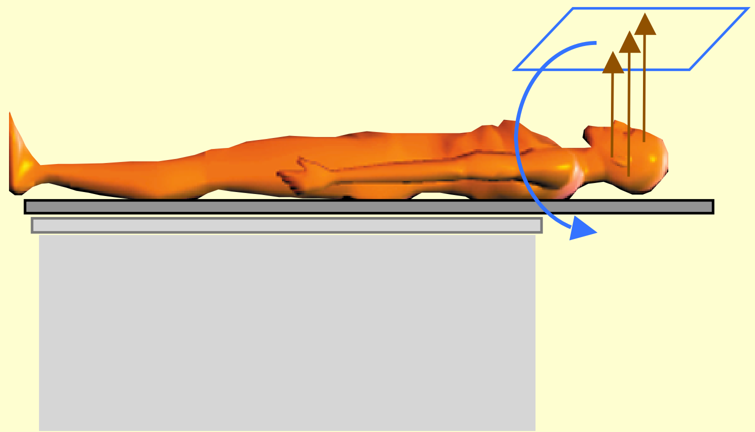
❶ Transmission tomography



- External radiation source
- Measurement of radiations transmitted through the patient
- Parameters related to the interactions of radiations within the body

e.g., Computed Tomography (CT)

❷ Emission tomography

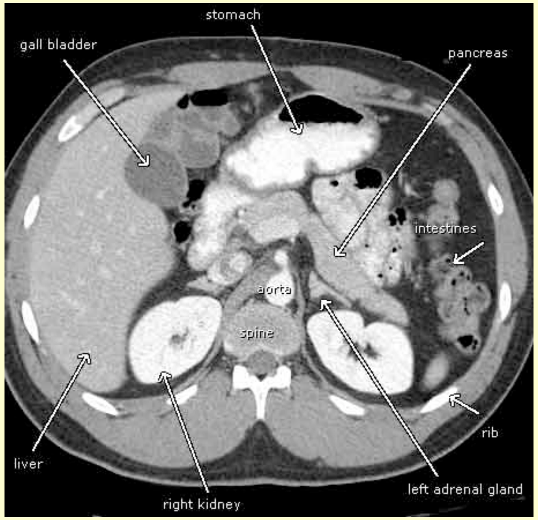


- Internal radiation source
- Measurement of radiation emitted from the patient
- Parameters related to the radiation sources within the body

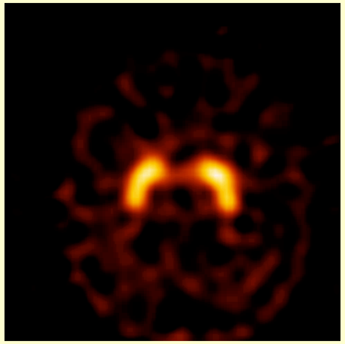
e.g., SPECT and PET

Different types of tomography

CT

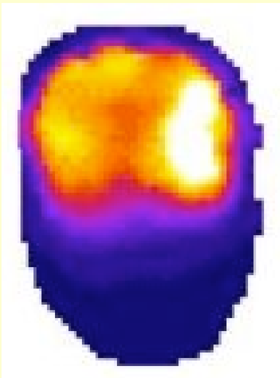


SPECT

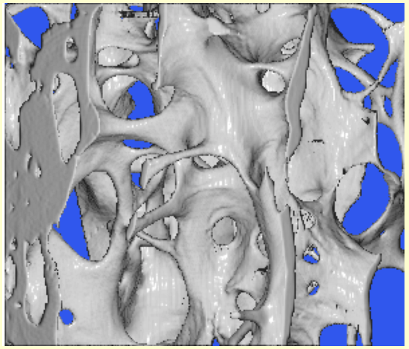


PET

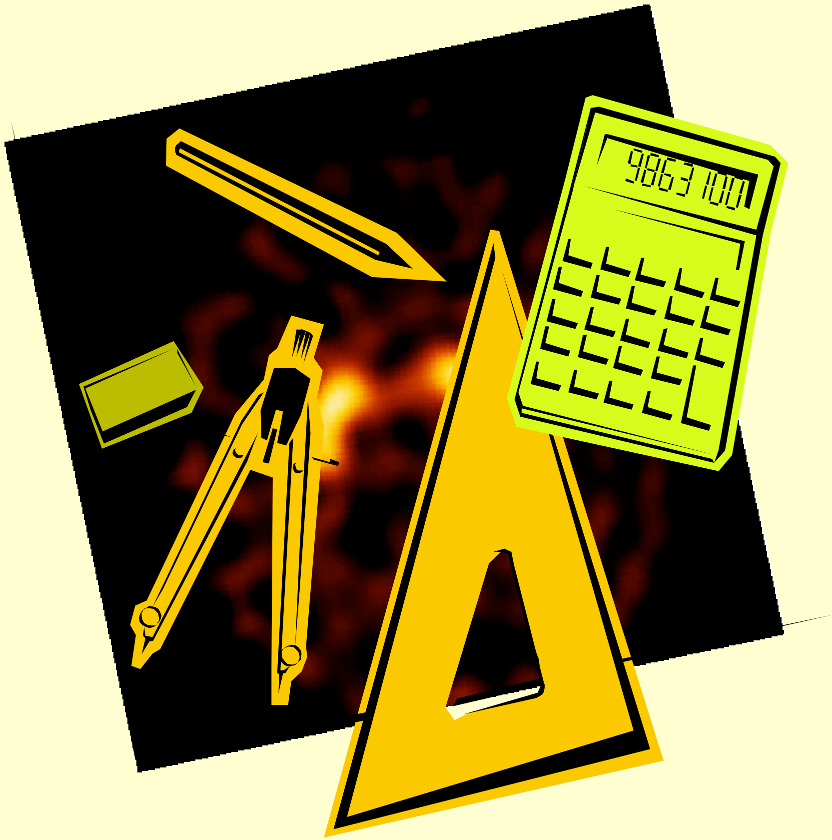
Optical Tomography



Synchrotron Tomography

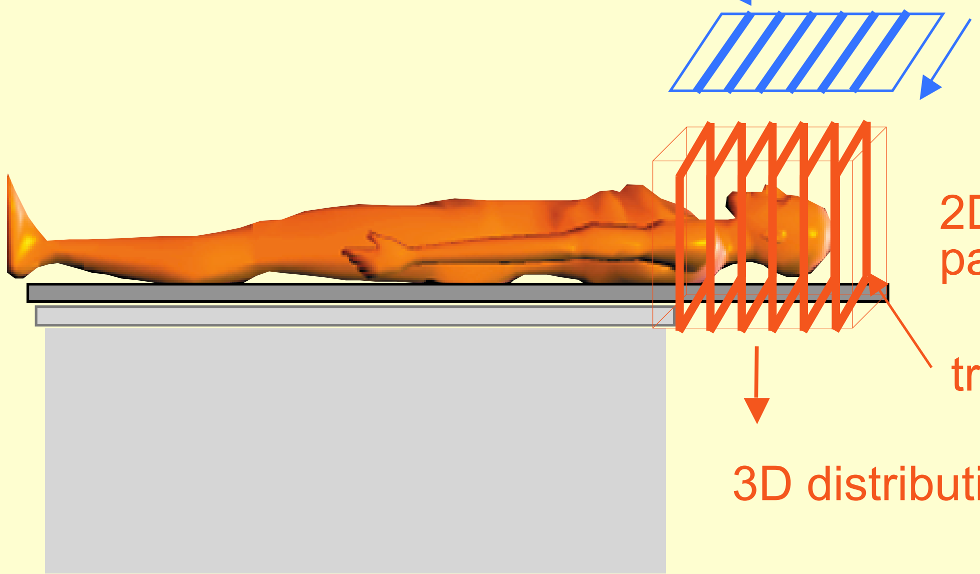


Basics



Factorization of the reconstruction problem

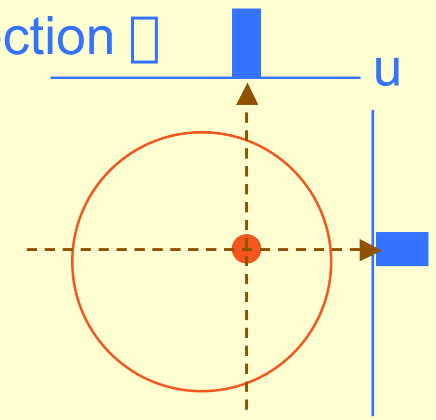
2D images from a set of 1D measurements

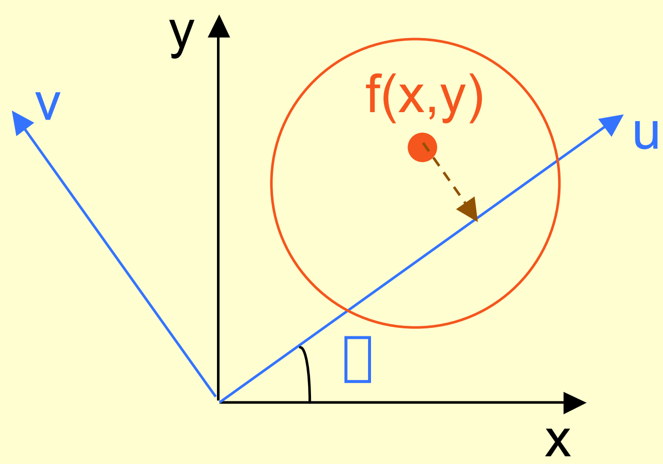


If real 3D: "Fully 3D reconstruction"

Key notion 1: projection

Modelling the direct problem



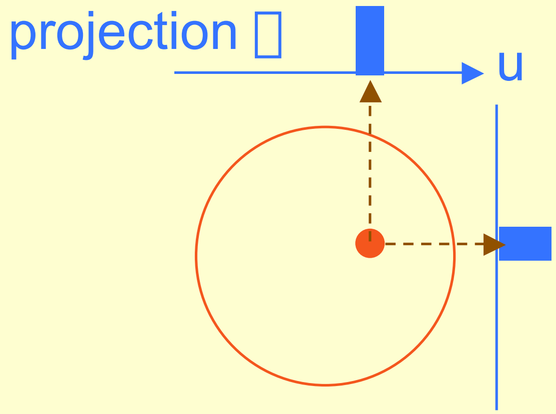


$$u = x\cos\varphi + y\sin\varphi$$
$$v = -x\sin\varphi + y\cos\varphi$$

$$p(u,\varphi) = \int_{-\infty}^{+\infty} f(x,y)\,dv$$

Projection: mathematical expression

The 2D Radon transform



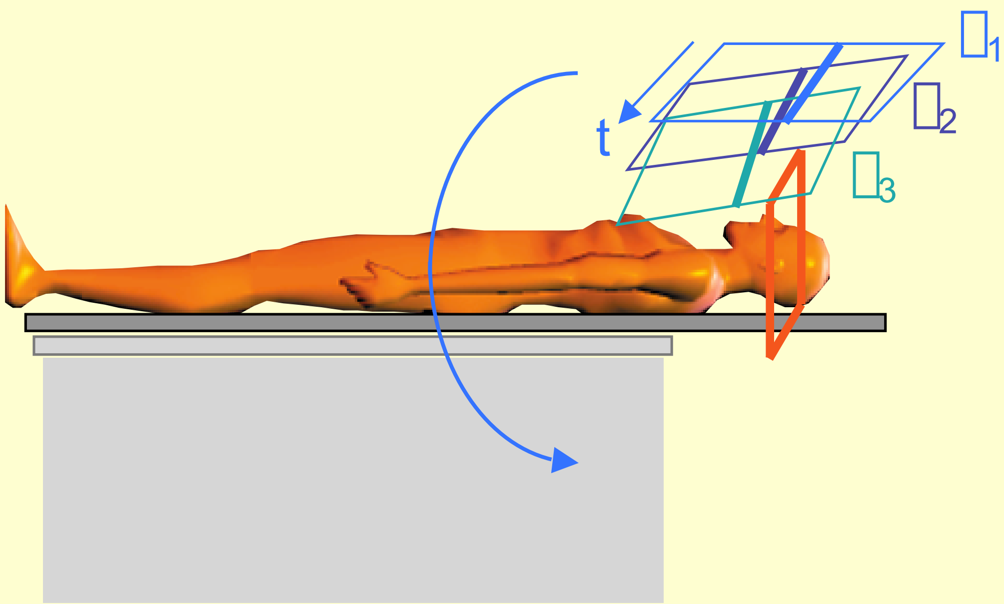
$$p(u,\theta) = \int_{-\infty}^{+\infty} f(x,y)\, dv$$

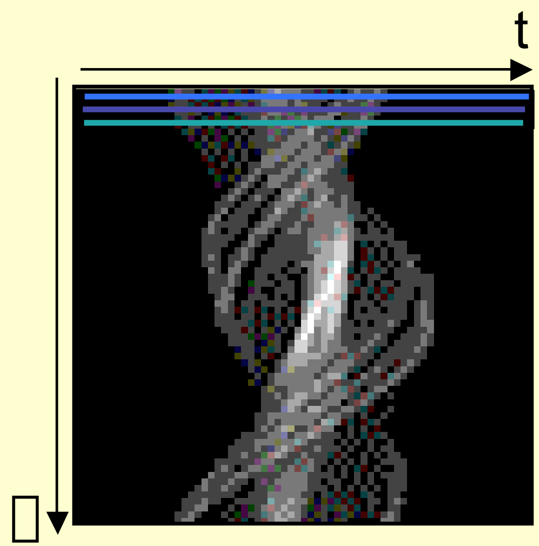
set of projections for $\theta = [0, \pi]$
= Radon transform of $f(x,y)$

$$R[f(x,y)] = \int_{0}^{\pi} p(u,\theta)\, d\theta$$

Key notion 2: sinogram

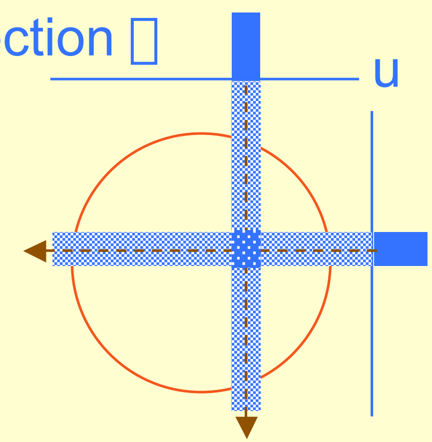
All detected signal concerning 1 slice

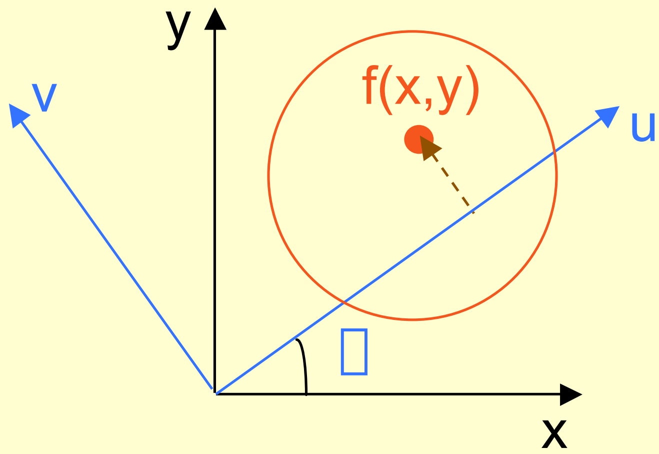




Key notion 3: backprojection

Tackling the inverse problem





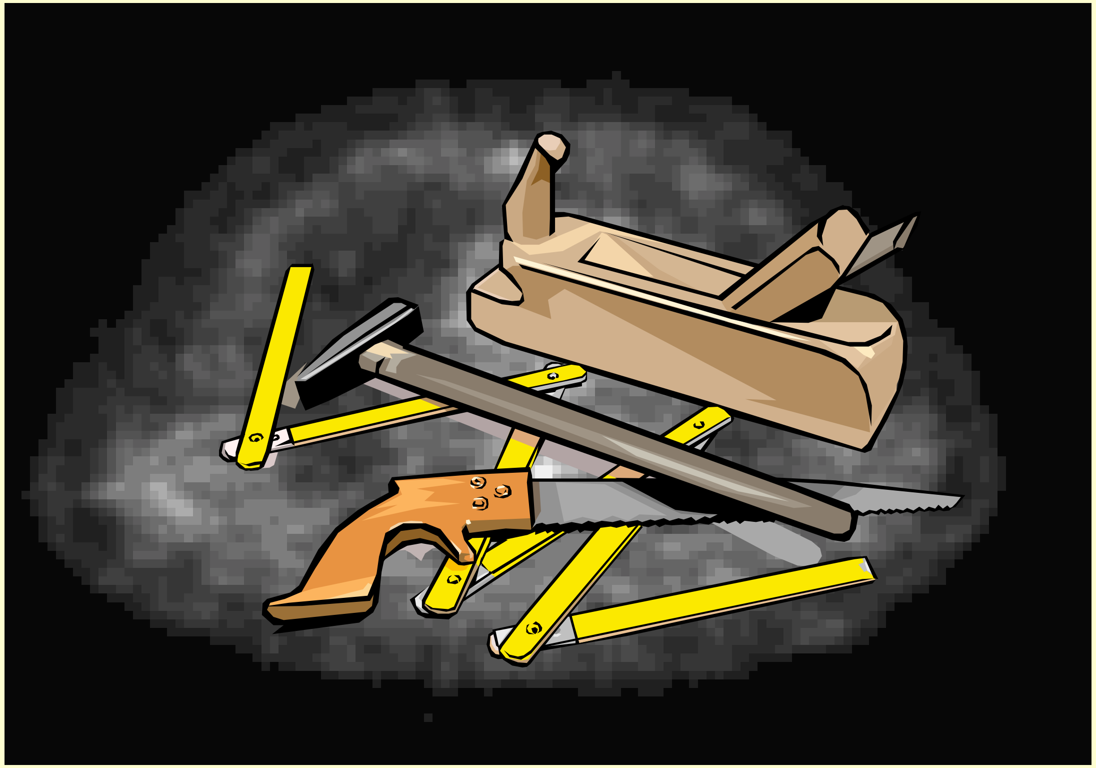
$u = x \cos\varphi + y \sin\varphi$

$v = -x \sin\varphi + y \cos\varphi$

$$f^*(x,y) = \int_0^{\pi} p(u,\varphi)\, d\varphi$$

Beware: backprojection is not the inverse of projection !

Methods of tomographic reconstruction



Two approaches

❶ Analytical approaches

$$f^*(x,y) = \int_0^{\pi} p'(u,\phi)\, d\phi$$

- Continuous formulation
- Explicit solution using inversion formulae or successive transformations
- Direct calculation of the solution
- Fast
- Discretization for numerical implementation only

❷ Discrete approaches

$$p_i = \sum_j r_{ij} f_j$$

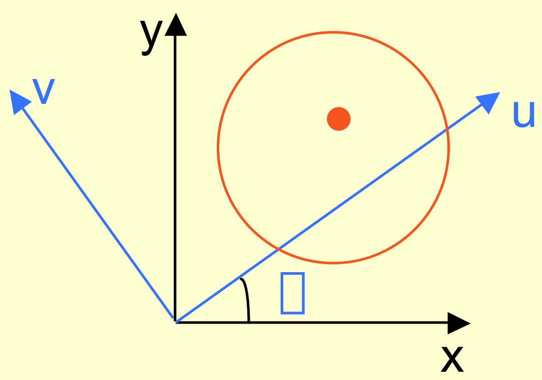
- Discrete formulation
- Resolution of a system of linear equations or probabilistic estimation
- Iterative algorithms
- Slow convergence
- Intrinsic discretization

Analytical approach: central slice theorem

Fourier transform

$p(u,\phi) \longrightarrow P(\nu,\phi) = \int_{-\infty}^{+\infty} p(u,\phi)\, e^{-i2\pi\nu u}\, du$

$p(u,\phi) = \int_{-\infty}^{+\infty} f(x,y)\, dv \qquad P(\nu,\phi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, e^{-i2\pi\nu u}\, du.dv$



$u = x \cos\phi + y \sin\phi$

$v = -x \sin\phi + y \cos\phi$

$\nu_x = \nu \cos\phi$

$\nu_y = \nu \sin\phi$

$du.dv = dx.dy$

$$P(\nu,\phi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, e^{-i2\pi(x\nu_x+y\nu_y)}\, dx.dy$$

$$P(\nu,\phi) = F(\nu_x,\nu_y)$$

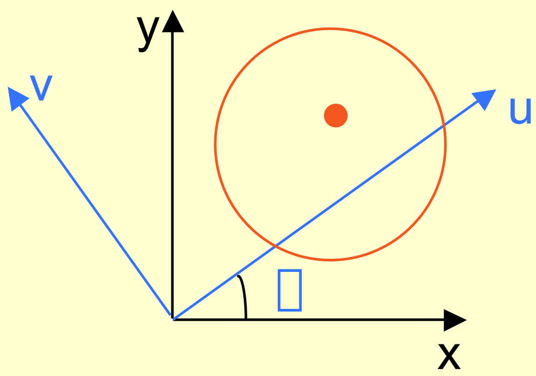
1D FT of p with respect to u = 2D FT of f in a specific direction

Analytical approach: filtered backprojection (FBP)

$$P(\rho,\phi) = F(\nu_x,\nu_y)$$

$$f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\nu_x,\nu_y)\, e^{i2\pi(x\nu_x+y\nu_y)}\, d\nu_x.d\nu_y$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(\rho,\phi)\, e^{i2\pi(x\nu_x+y\nu_y)}\, d\nu_x.d\nu_y$$



$u = x\cos\phi + y\sin\phi$

$\nu_x = \rho\cos\phi$

$\nu_y = \rho\sin\phi$

$\rho = (\nu_x{}^2 + \nu_y{}^2)^{1/2}$

$d\nu_x d\nu_y = |\rho| d\rho d\phi$

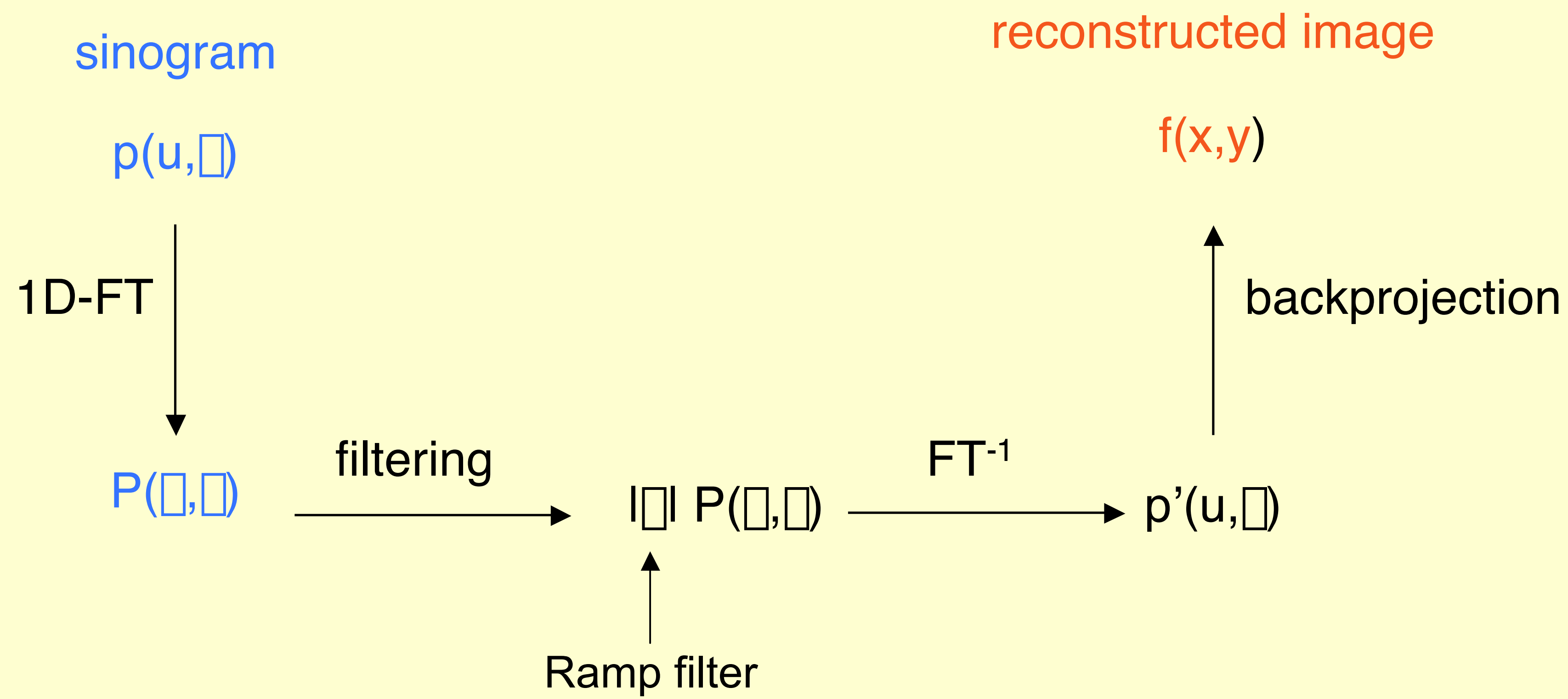
$$= \int_0^{\pi}\int_{-\infty}^{+\infty} P(\rho,\phi)\, |\rho| e^{i2\pi\rho u}\, d\rho.\, d\phi$$

$$= \int_0^{\pi} p'(u,\phi)\, d\phi \quad \text{with} \quad p'(u,\phi) = \int_{-\infty}^{+\infty} P(\rho,\phi)\, |\rho| e^{i2\pi\rho u}\, d\rho$$

Ramp filter

Filtered backprojection: algorithm

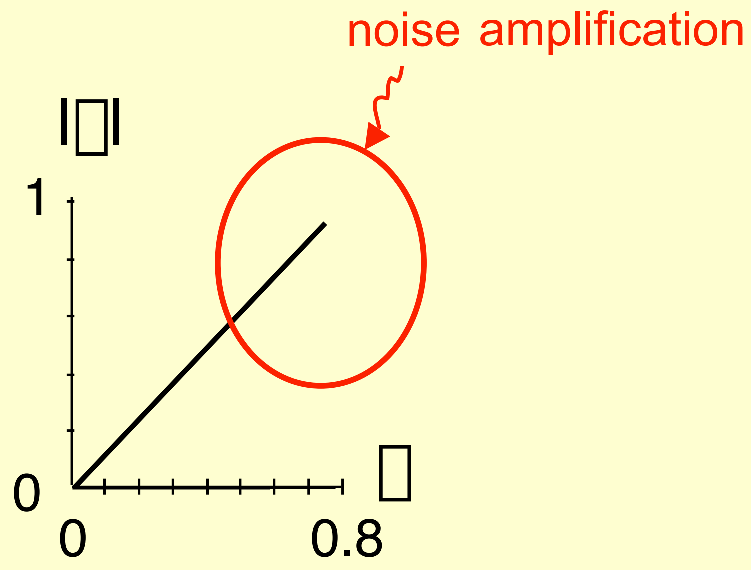
$$f(x,y) = \int_0^{\pi} p'(u,\varphi)\, d\varphi \qquad \text{with} \qquad p'(u,\varphi) = \int_{-\infty}^{+\infty} P(\nu,\varphi)\, |\nu|\, e^{i2\pi\nu u}\, d\nu$$

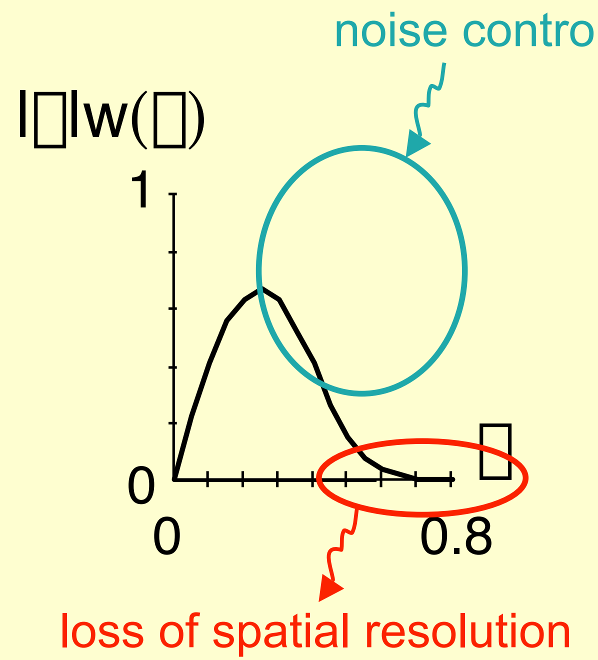


Filtered backprojection: beyond the Ramp filter

$$f(x,y) = \int_0^{\pi} p'(u,\varphi)\, d\varphi \quad \text{with} \quad p'(u,\varphi) = \int_{-\infty}^{+\infty} P(\nu,\varphi)\, |\nu| e^{i2\pi\nu u}\, d\nu$$

$|\nu| \rightarrow |\nu| w(\nu)$





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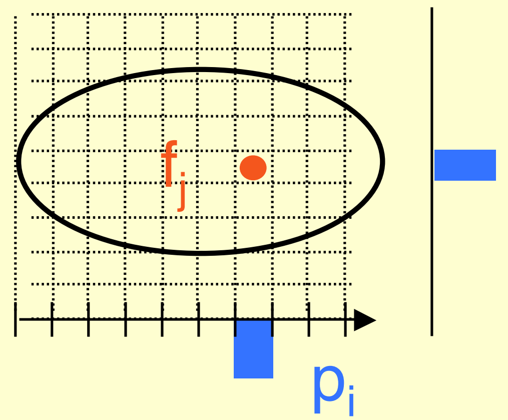
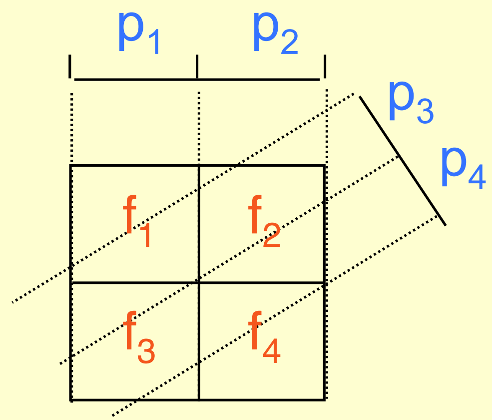
- Iterative algorithms
- Slow convergence
- Intrinsic discretization

Discrete approach: model





$$
\begin{aligned}
p_1 &= r_{11}f_1 + r_{12}f_2 + r_{13}f_3 + r_{14}f_4 \\
p_2 &= r_{21}f_1 + r_{22}f_2 + r_{23}f_3 + r_{24}f_4 \\
p_3 &= r_{31}f_1 + r_{32}f_2 + r_{33}f_3 + r_{34}f_4 \\
p_4 &= r_{41}f_1 + r_{42}f_2 + r_{43}f_3 + r_{44}f_4
\end{aligned}
$$

$$
\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} r_{11} & \cdots & r_{14} \\ \vdots & \ddots & \vdots \\ r_{41} & \cdots & r_{44} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}
$$

$$p = R\,f$$

↑ projector

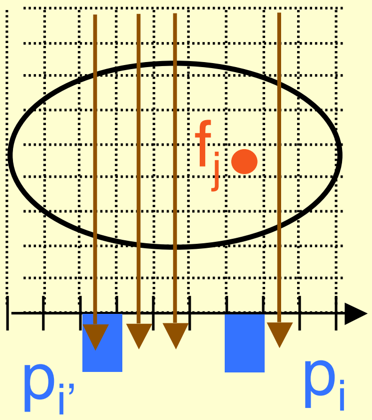
Given p and R, estimate f

Discrete approach: calculation of R

$p = R\, f$ R models the direct problem

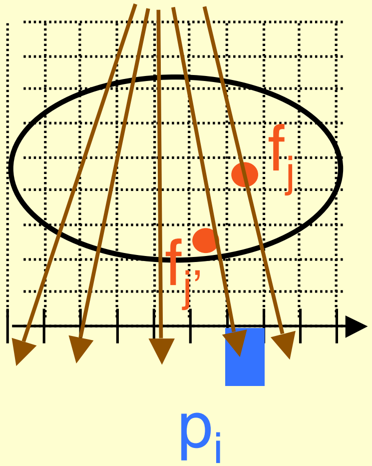
- Geometric modelling
  - intersection between pixel and projection rays

parallel



$r_{ij}=1$
$r_{i'j}=0$

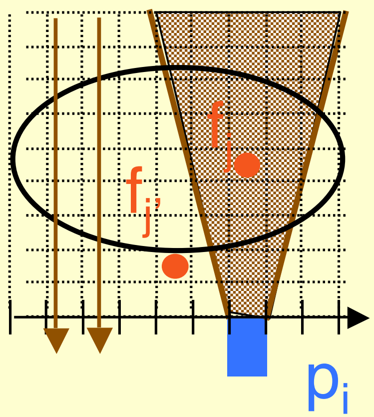
fan beam



$r_{ij}=0$
$r_{ij'}=1$

- Physics modelling
  - spatial resolution of the detector
  - physical interactions of radiations

spatial resolution



$r_{ij}\neq 0$, j
$r_{ij'}=0$, j

Two classes of discrete methods

❶ Algebraic methods

❷ Statistical approaches

$$p_i = \sum_j r_{ij} f_j$$

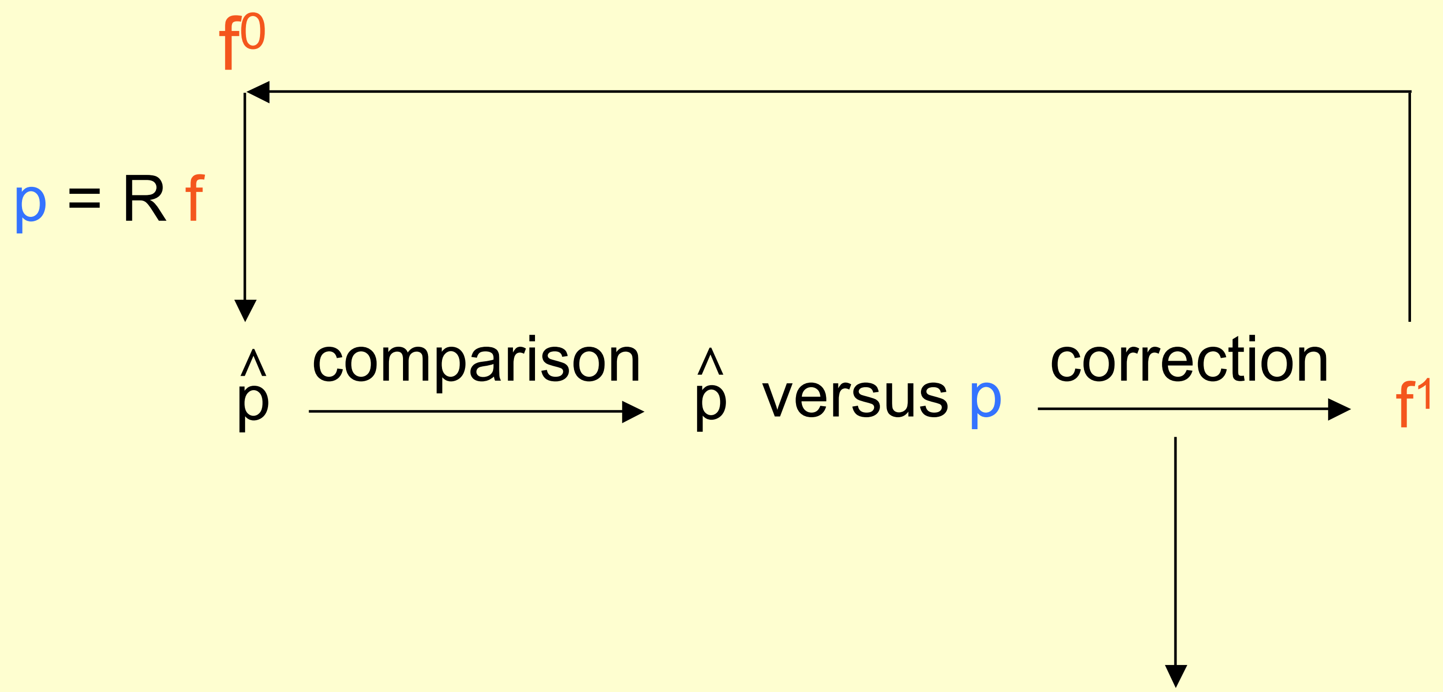
- Generalized inverse methods

- Bayesian estimates
- Optimization of functionals

- Account for noise properties

Iterative algorithm used in discrete methods

$$p = R\,f$$



defines the iterative method:

additive if $f^{n+1} = f^{n} + c^{n}$

multiplicative if $f^{n+1} = f^{n} \, . \, c^{n}$

Algebraic methods

$$p = R\,f$$

Minimisation of $||p - R\,f||^2$

Several minimisation algorithms are possible to estimate a solution:

e.g., SIRT (Simultaneous Iterative Reconstruction Technique)
Conjugate Gradient
ART (Algrebraic Reconstruction Technique)

e.g., additive ART:

$$f_j^{n+1} = f_j^{n} + (p_i - p_i^{n})\, r_{ij} / \sum_k r_{ik}^2$$

Statistical methods

$$p = R\,f$$

Probabilistic formulation (Bayes' equation):

$$proba(f|p) = proba(p|f)\ proba(f) \,/\, proba(p)$$

- $proba(f|p)$: probability of obtaining f when p is measured
- $proba(p|f)$: likelihood of p
- $proba(f)$: prior on f
- $proba(p)$: prior on p

Find a solution f maximizing *proba*(p|f) given a probabilistic model for p

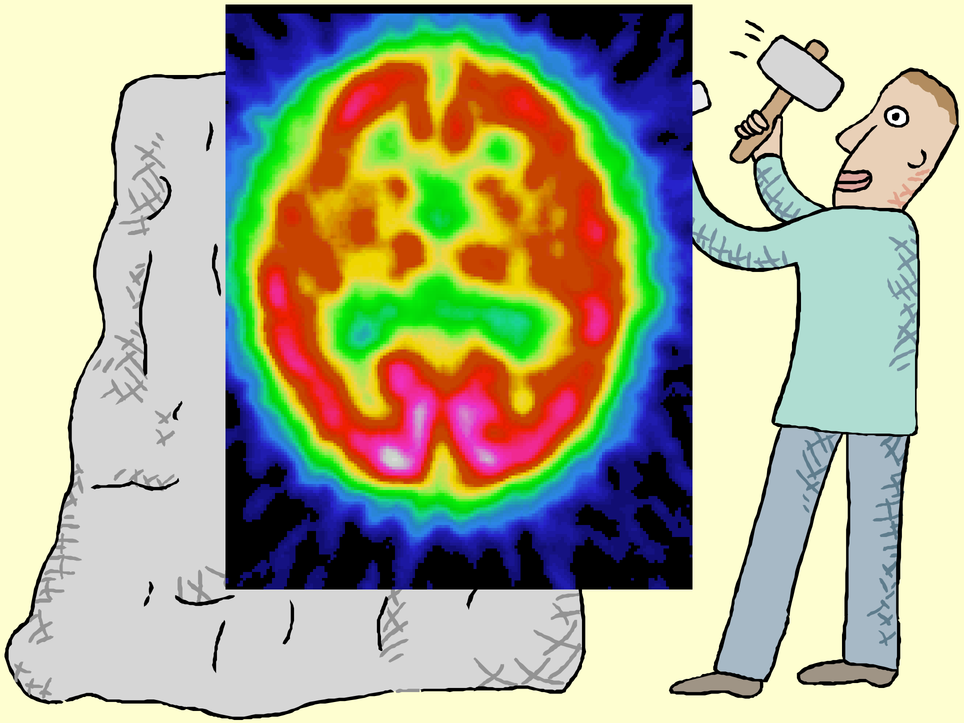
e.g., if p follows a Poisson law: $proba(p|f) = \prod_k \exp(-\bar{p}_k).\bar{p}_k^{\,p_k} / p_k!$

MLEM (Maximum Likelihood Expectation Maximisation):

$$f^{n+1} = f^n \cdot R^t(p/p^n)$$

and OSEM (accelerated version of MLEM)

Regularization



Regularization

Set constraints on the solution f

Solution f:

trade-off between

the agreement with the observed data

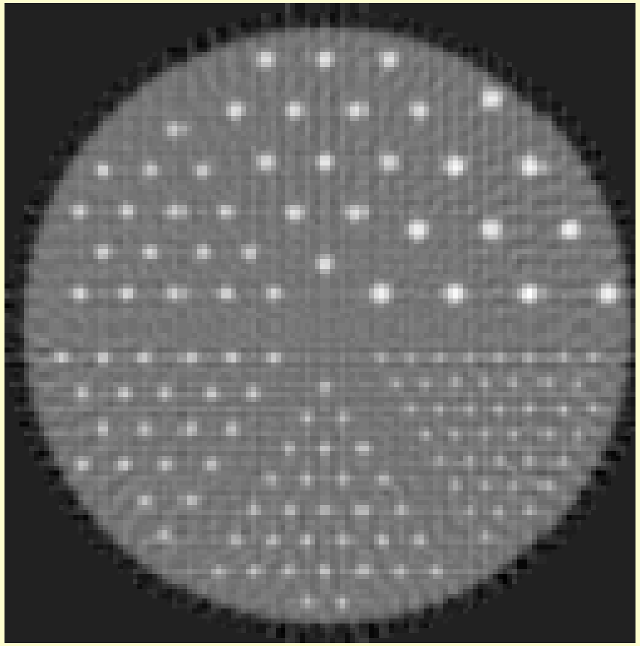
and

the agreement with the constraints

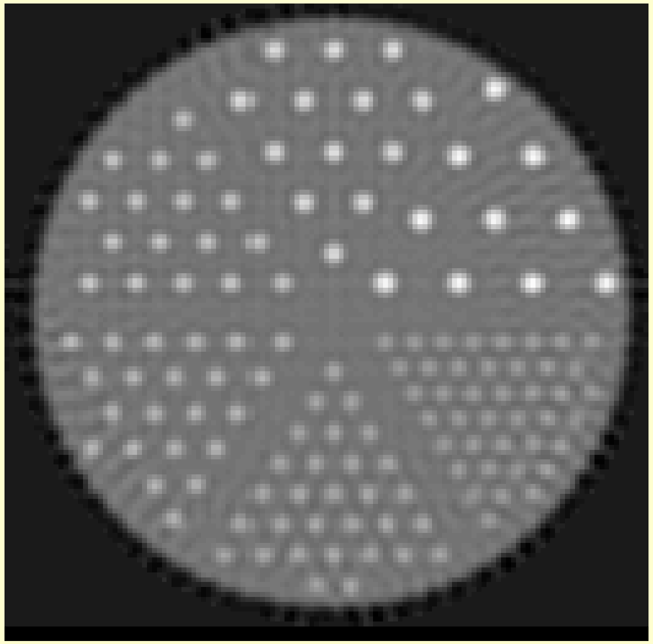
Regularization for analytical methods

Filtering

$$f(x,y) = \int_0^{\pi}\int_{-\infty}^{+\infty} P(\nu,\phi)\, w(\nu)\, |\nu|\, e^{i2\pi\nu u}\, d\nu$$



Ramp filter



Butterworth filter

Regularization for discrete methods

Minimisation of $||p - R\,f||^2 + □\,K(f)$

□ controls the trade-off between
agreement with the projections and agreement with the constraints

$$proba(f|p) = proba(p|f)\ proba(f)\ /\ proba(p)$$

↑ prior on f, i.e. proba(f) non uniform

Examples of priors:

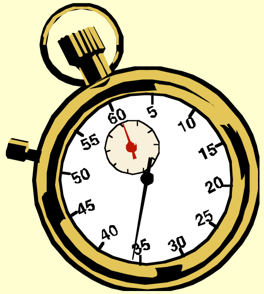
- f smooth
- f having discontinuities

Conjugate Gradient gives MAP-Conjugate Gradient (Maximum A Posteriori)
MLEM gives MAP-EM

New challenges

Analytical reconstruction

- Fully 3D reconstruction for CT:
  - suited to the new designs of CT detectors (increased number of rows, increased gantry rotation speed, helical cone beam geometry)
  - real time



- Reconstruction algorithms managing source deformations (motions)

Iterative reconstruction: fully 3D Monte Carlo reconstruction

$$p = R\, f$$

modelling R using numerical (Monte Carlo) simulations
of the imaging procedure in emission tomography

→

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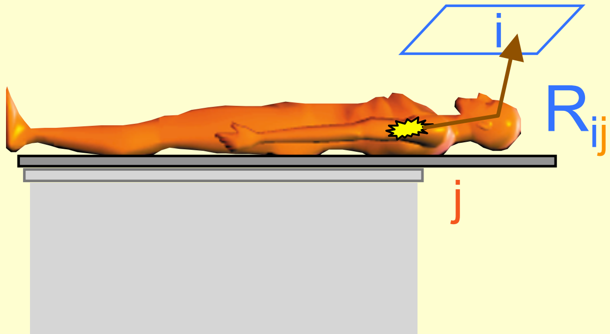

tissue density
and composition
+
cross-section tables of
radiation interaction

stochastic modelling
of physical interactions

probability that an
"event" in j be detected
in i

All propagation and detection physics can be accurately modelled
3D propagation of radiation is taken into account

Pending issues

- Size of the fully 3D problem:
  64 projections 64 x 64, R includes 64^6 elements (256 Gigabytes)

- Time

Question 1

Given 128 projections of 64 pixels by 64 pixels, you can easily create:

A. 64 sinograms with 128 rows and 128 columns

B. 128 sinograms with 64 rows and 64 columns

C. 64 sinograms with 128 rows and 64 columns

D. 64 sinograms with 64 rows and 128 columns

E. 128 sinograms with 128 rows and 64 columns

Question 2

Iterative reconstruction:

A. Is faster than analytical reconstruction

B. Is necessarily fully 3D

C. Always involves a regularization term

D. Is based on a discrete formalism

E. Can include accurate modelling of the radiation physics