# Principles of PET: reconstruction / simulations

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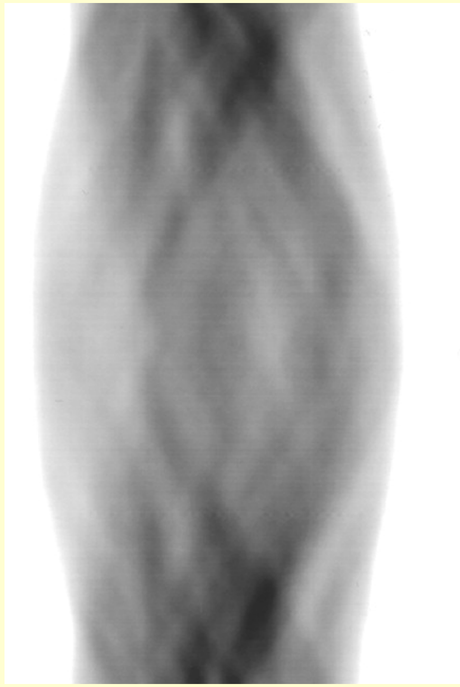
# Outline

- Why does PET require tomographic reconstruction?
- Basics
- Tomographic reconstruction methods
- From tomographic reconstruction to Monte Carlo simulations
- Conclusions

# Why tomographic reconstruction ? (1)

- What does a PET system detect?



events stored as sinograms

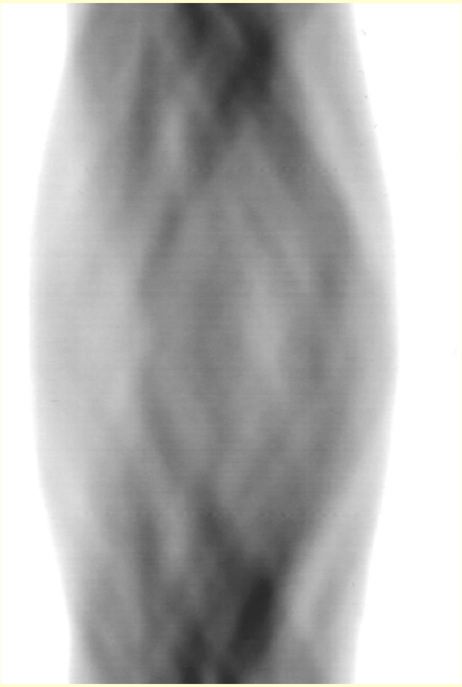
$$LOR\ i_1, E_1, E'_1, t_1, LOR\ i_2, E_2, E'_2, t_2, \ldots$$

a list of events (list mode)

Do you recognize a brain ?

# Why tomographic reconstruction ? (2)

- How to get meaningful images ?



a sinogram

?

$$LOR\ i_1, E_1, E'_1, t_1, LOR\ i_2, E_2, E'_2, t_2, \ldots$$

a list of events (list mode)

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# Why tomographic reconstruction ? (2)

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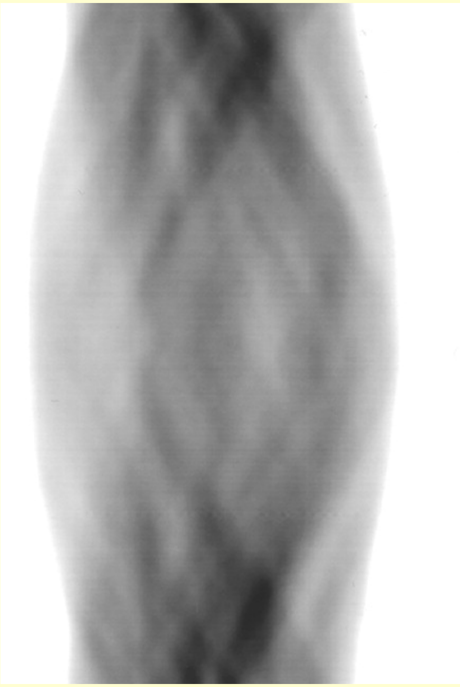


a sinogram

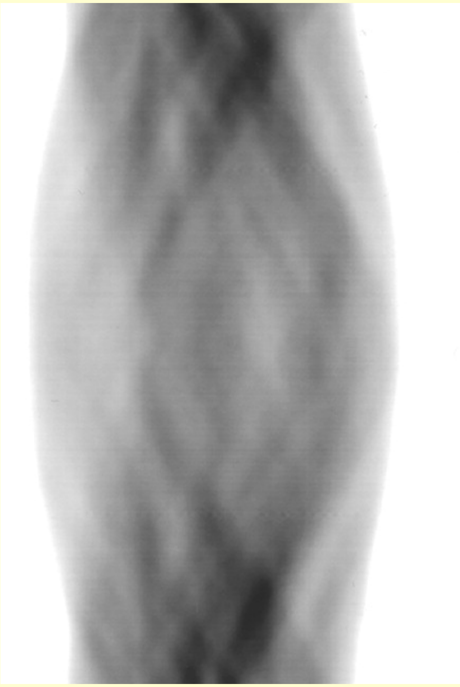
using tomographic reconstruction

$LOR\ i_1, E_1, E'_1, t_1, LOR\ i_2, E_2, E'_2, t_2, \ldots$

a list of events (list mode)

# Toward tomographic reconstruction

- To be able to go from the sinogram (or list mode) to the image…



a sinogram

using tomographic reconstruction

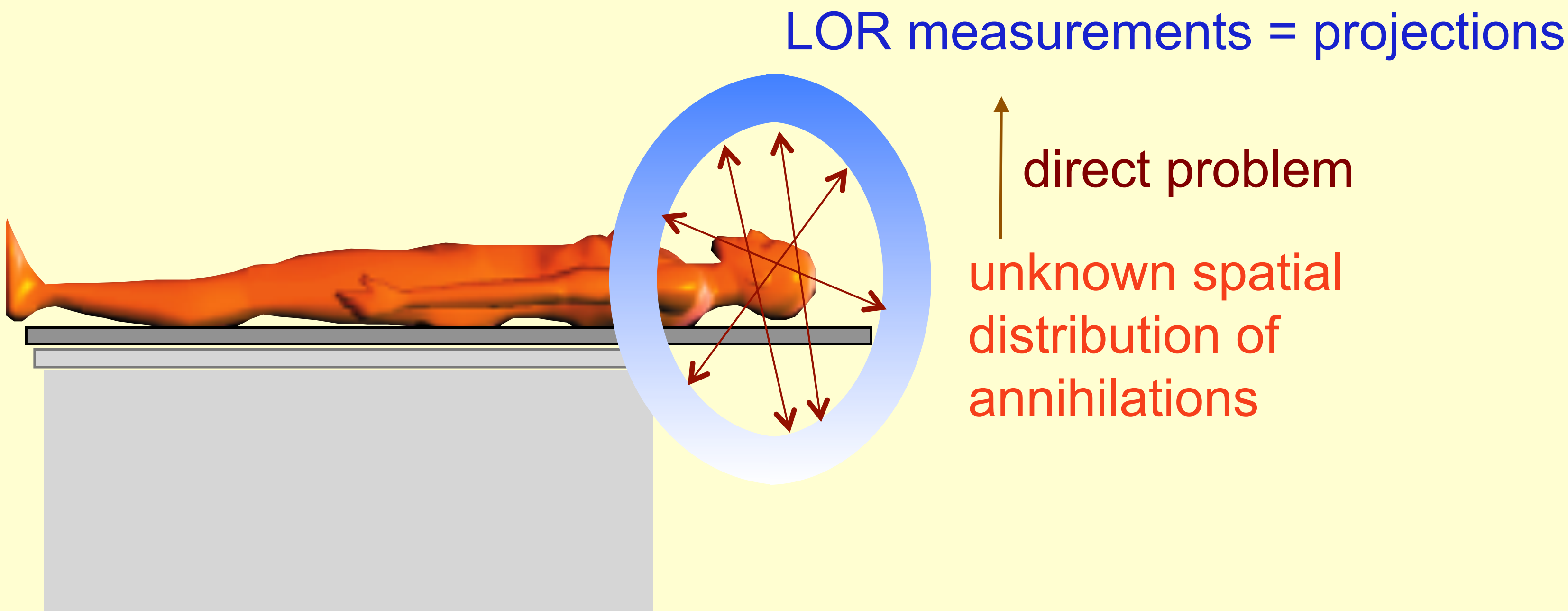
$LOR\ i_1, E_1, E'_1, t_1, LOR\ i_2, E_2, E'_2, t_2, \ldots$

a list of events (list mode)

… one first have to understand how the object produces a sinogram

# The direct (forward) problem in PET

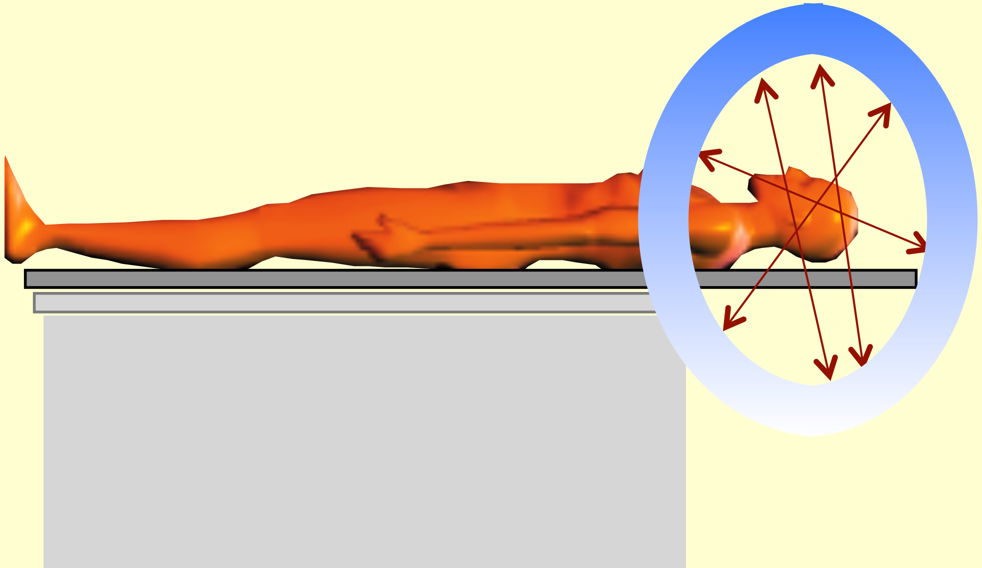
The PET detector measures a set of “projection” data: integrals of annihilations along certain directions, called Lines of Response (LOR).




The mathematical formulation of the relationship between the unknown parameters and the measurements is the direct problem.

# Inverse problem

Tomographic reconstruction is the inversion of the direct problem.



LOR measurements = projections

inverse problem

unknown spatial distribution of annihilations

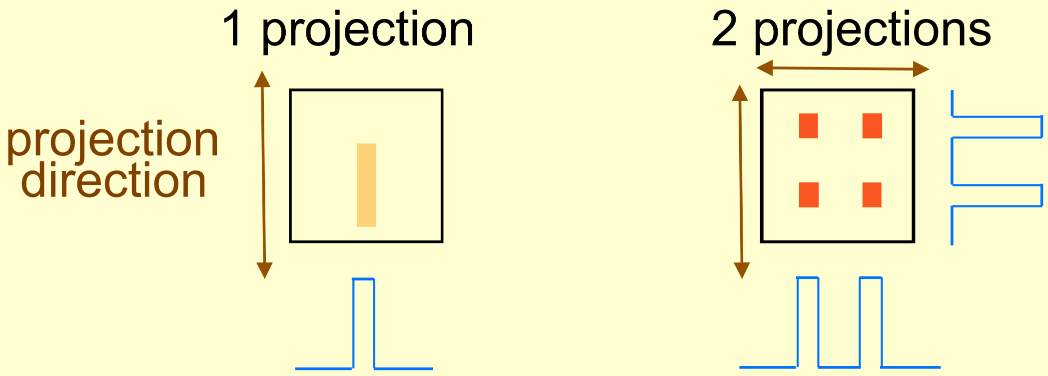
Inverse problem: estimating the 3D map of the annihilation points from the measured data.

# Is that easy ? No, still an active area of research since the mid 50's

*La leçon difficile, William Bouguereau (1825 - 1905)*

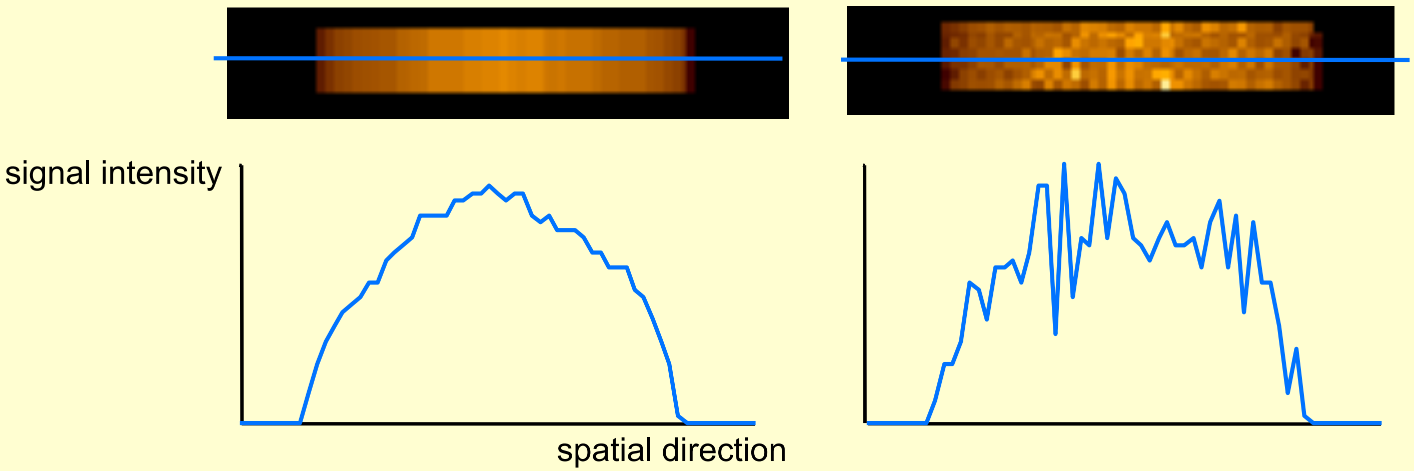
# Why so difficult ? An ill-posed inverse problem

- Limited angular sampling




… several solutions for any finite number of projections

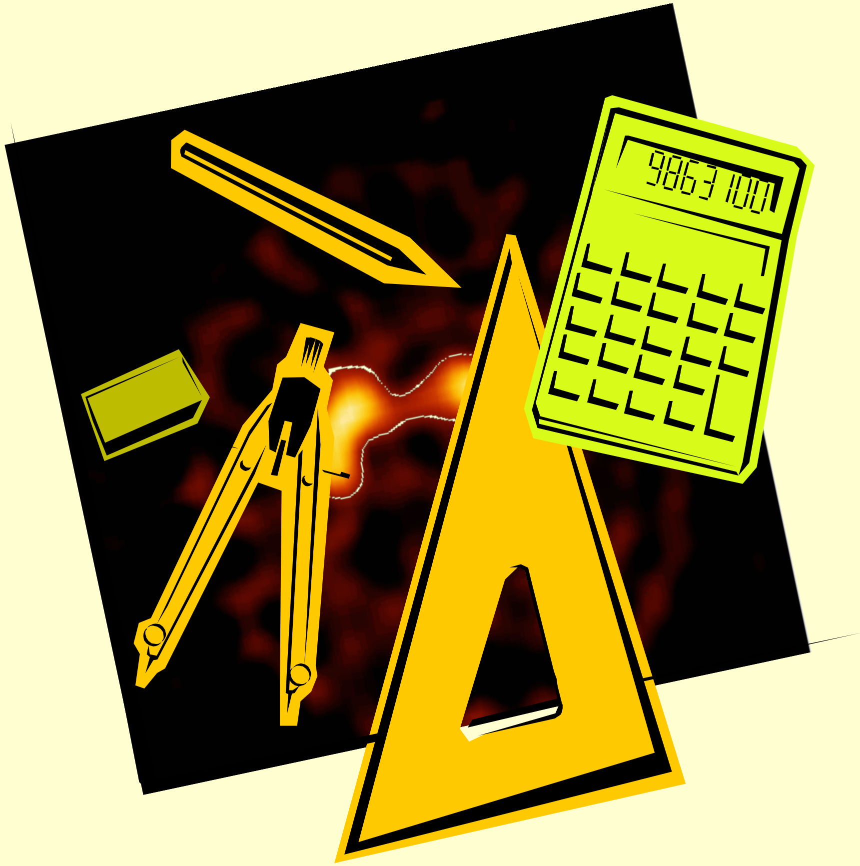
- Measurements are noisy




No noise Noisy

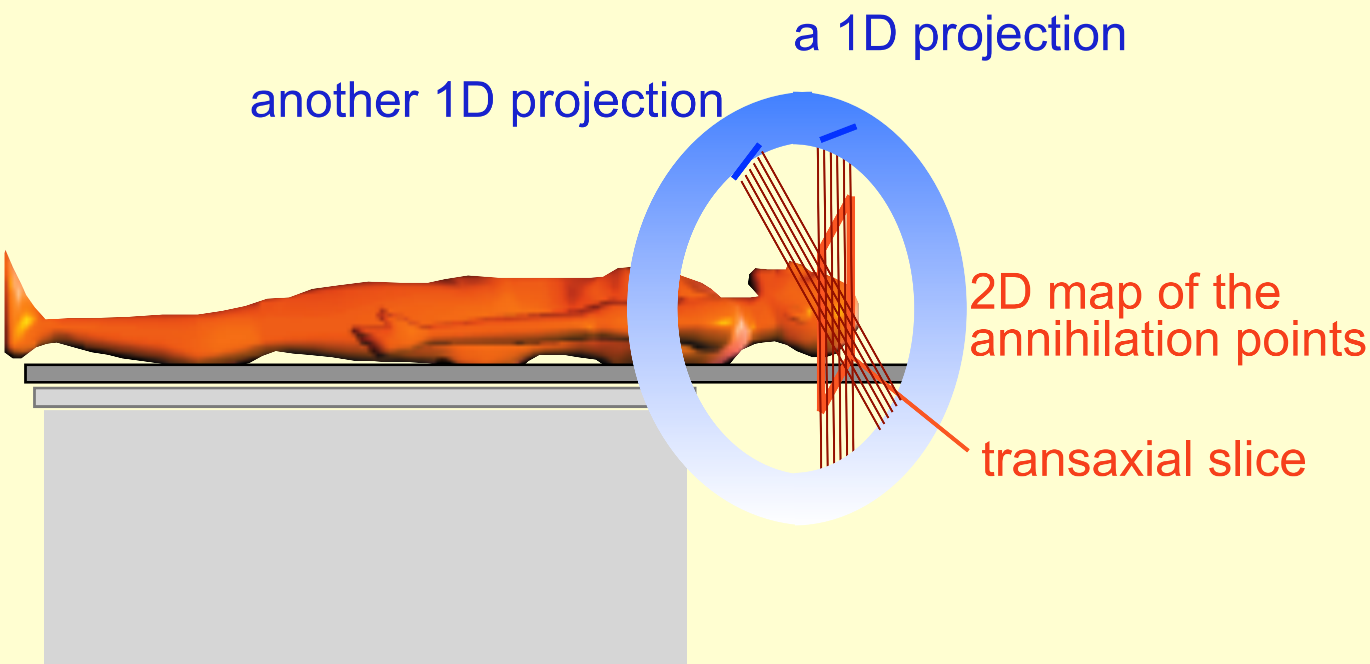
Ill-posed: no unique and accurate solutions,
several solutions compatible with the measurements

# Basics



# Simplified approach: factorization of the reconstruction problem

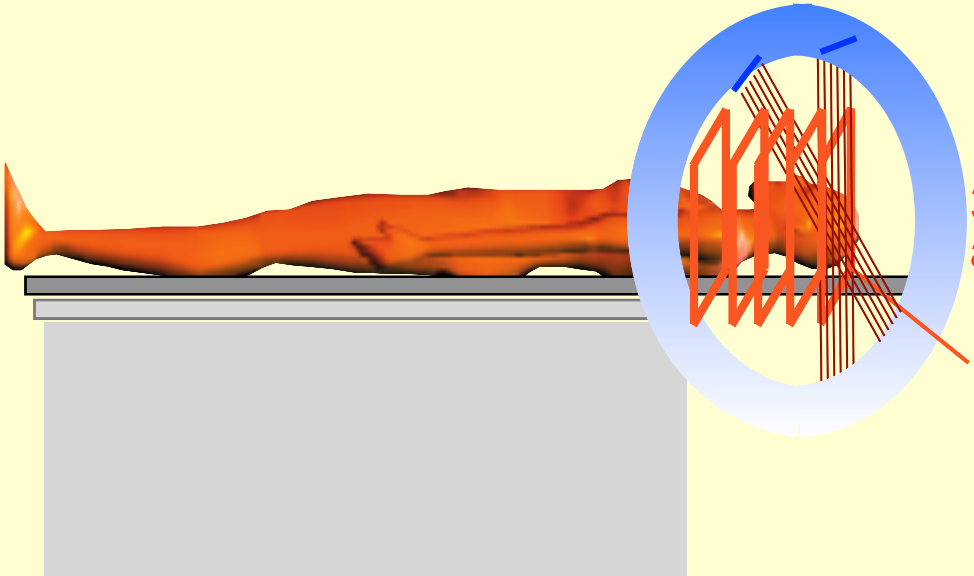
Reconstructing 2D images from a set of 1D measurements




a 1D projection is a set of parallel LOR

# Simplified approach: factorization of the reconstruction problem

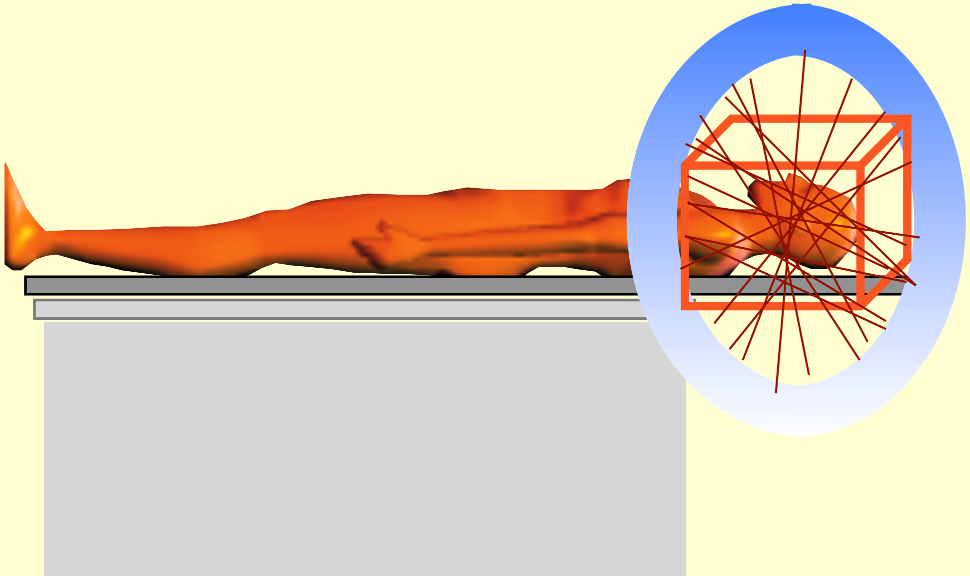
Then repeating this for all slices and stacking the slices to get a 3D volume




"Fake" 3D reconstruction as it is actually a set of 2D reconstructions

# State of the art is now fully 3D reconstruction

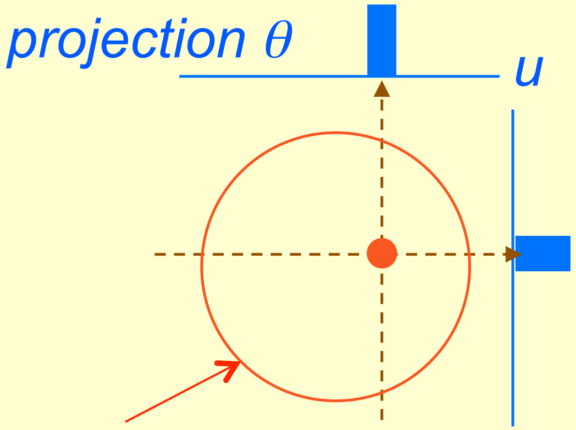
3D images from a set of 2D measurements



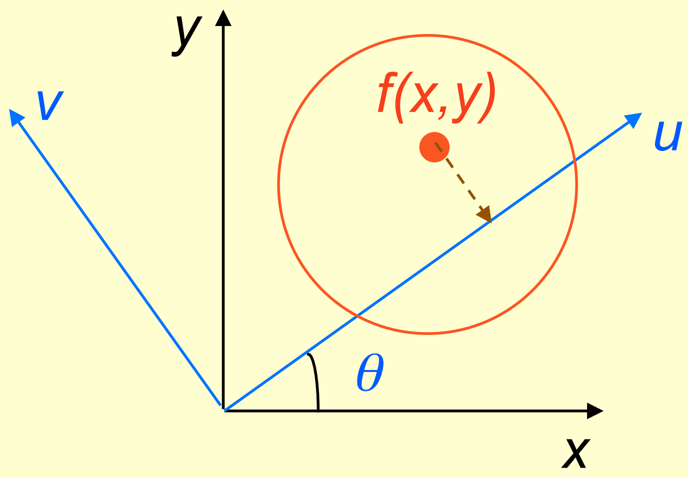

Still, for educational purpose, explaining reconstruction in 2D is easier

# Key notion 1: projection

## Modelling the direct problem




*This is the transaxial slice to be reconstructed*

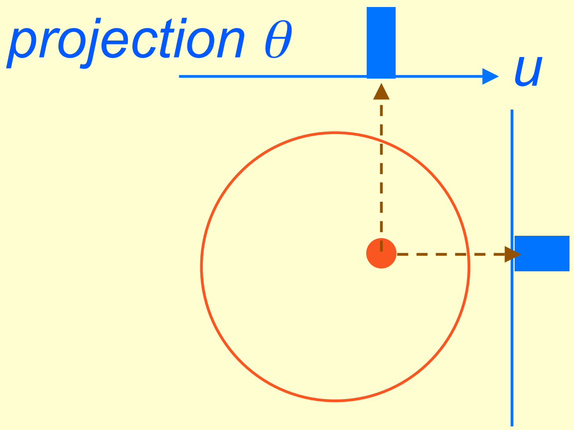



$$u = x \cos\theta + y \sin\theta$$
$$v = -x \sin\theta + y \cos\theta$$

$$p(u,\theta) = \int_{-\infty}^{+\infty} f(x,y)\, dv$$

# Projection: mathematical expression

## The 2D Radon transform




$$p(u,\theta) = \int_{-\infty}^{+\infty} f(x,y)\, dv$$

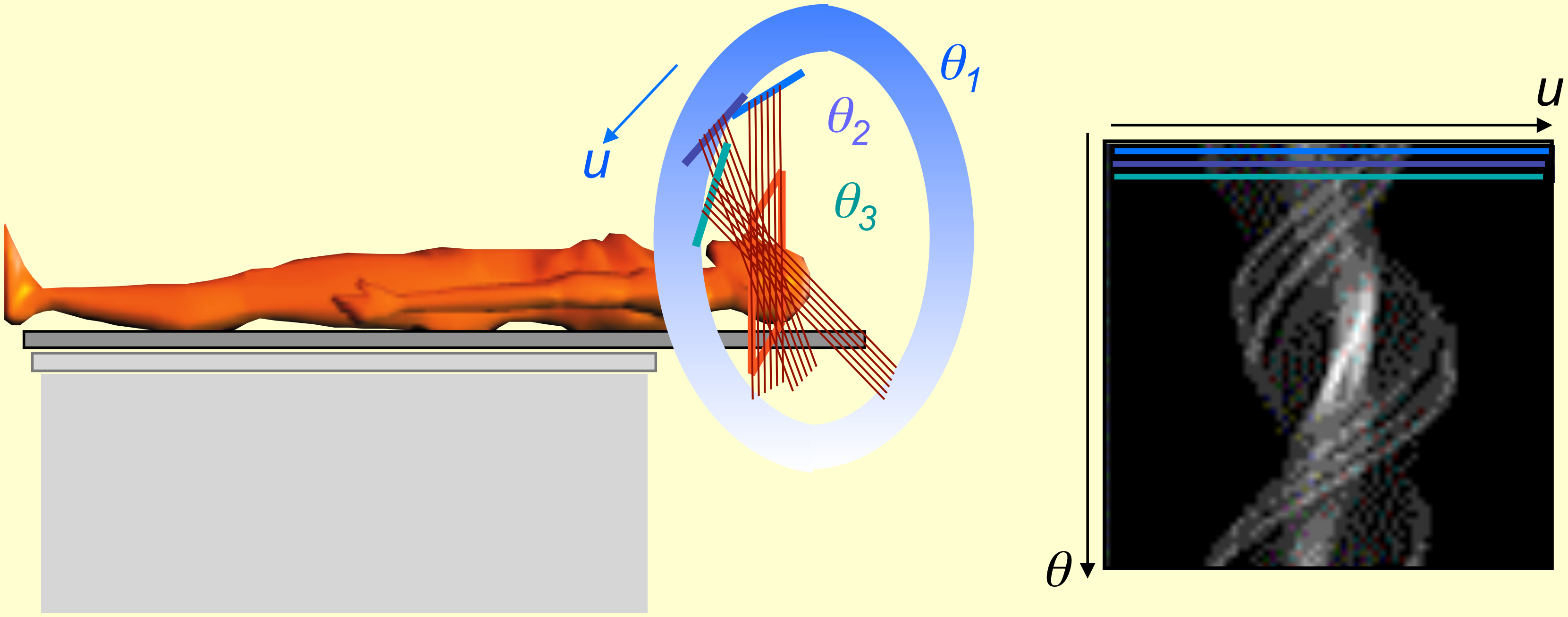
*set of projections for* $\theta = [0, \pi]$

*= Radon transform of* $f(x,y)$

$$R[f(x,y)] = \int_{0}^{\pi} p(u,\theta)\, d\theta$$

# Key notion 2: sinogram

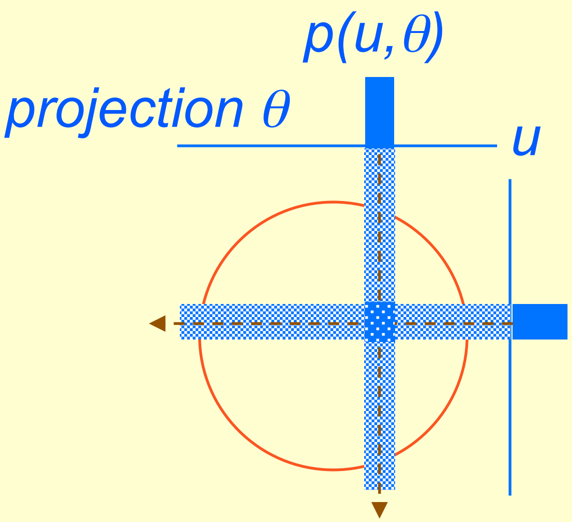
All detected signal associated with 1 slice

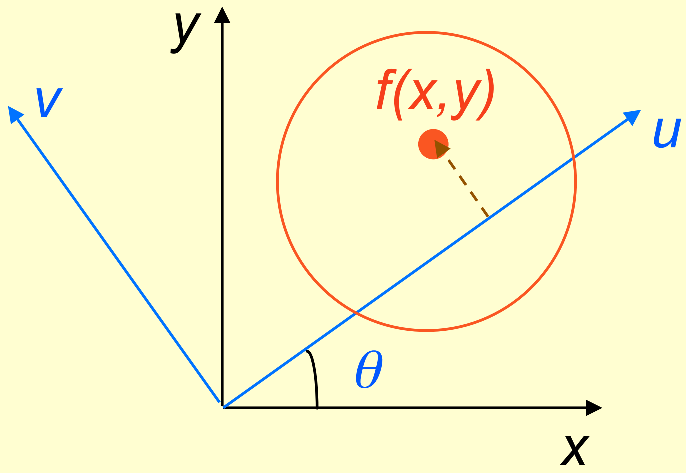



The sinogram is the Radon transform of the image

# Key notion 3: backprojection

## Tackling the inverse problem



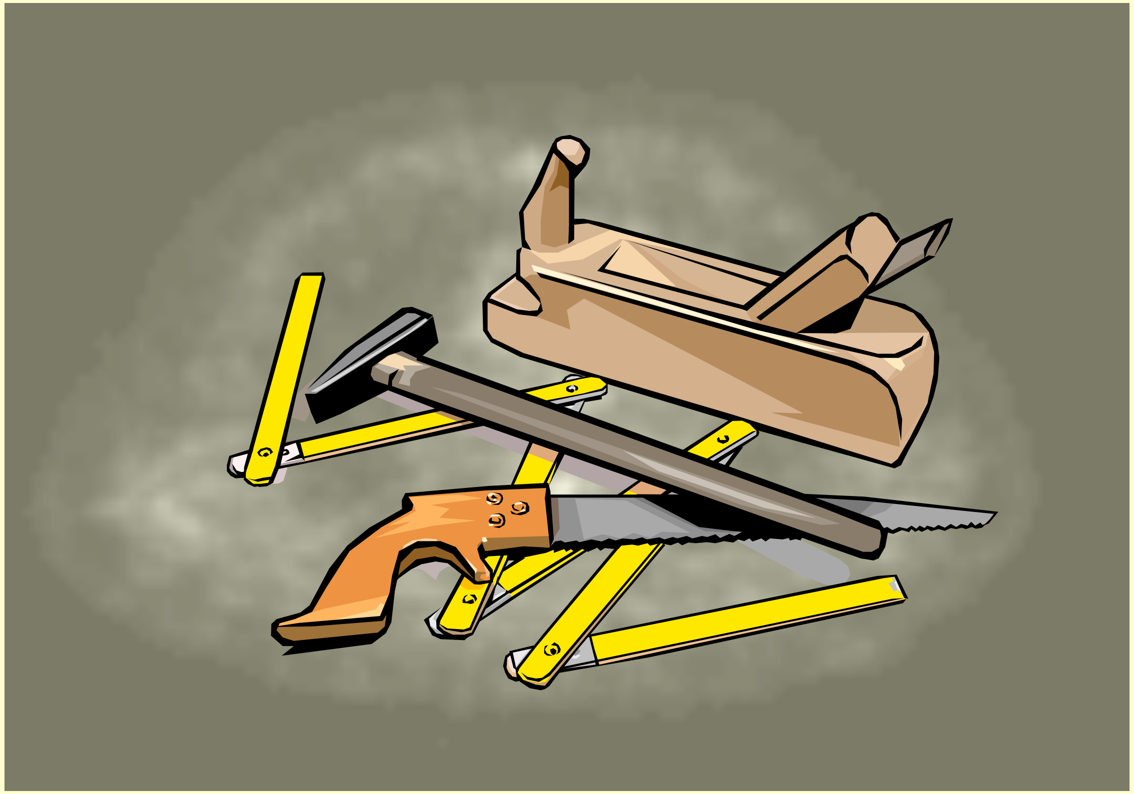

$u = x \cos\theta + y \sin\theta$

$v = -x \sin\theta + y \cos\theta$

$$f^*(x,y) = \int_0^{\pi} p(u,\theta)\, d\theta$$

Beware: backprojection is not the inverse of projection !

# Reconstruction methods



# Two approaches

❶ Analytical approaches

$$f^*(x,y) = \int_0^{\pi} p'(u,\theta)\, d\theta$$

- Continuous formulation
- Explicit solution using inversion formulae or successive transformations
- Direct calculation of the solution
- Fast
- Discretization for numerical implementation only

❷ Discrete approaches

$$p_i = \sum_j r_{ij} f_j$$

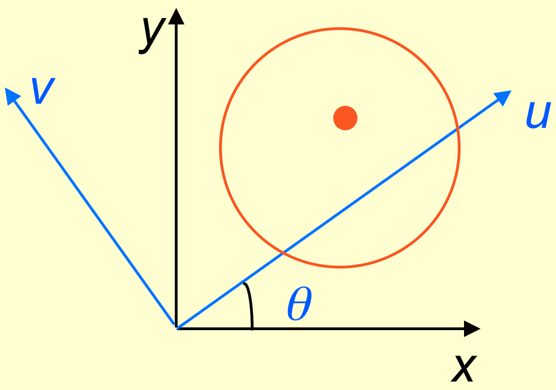
- Discrete formulation
- Resolution of a system of linear equations or probabilistic estimation
- Iterative algorithms
- Slow convergence
- Intrinsic discretization

# Analytical approach: central slice theorem

Fourier transform

$$p(u,\theta) \longrightarrow P(\rho,\theta) = \int_{-\infty}^{+\infty} p(u,\theta)\, e^{-i2\pi\rho u}\, du$$

$$p(u,\theta) = \int_{-\infty}^{+\infty} f(x,y)\, dv \qquad P(\rho,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, e^{-i2\pi\rho u}\, du.dv$$




$u = x \cos\theta + y \sin\theta$
$v = -x \sin\theta + y \cos\theta$
$\rho_x = \rho \cos\theta$
$\rho_y = \rho \sin\theta$
$du.dv = dx.dy$

$$P(\rho,\theta) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\, e^{-i2\pi(x\rho_x+y\rho_y)}\, dx.dy$$

$$P(\rho,\theta) = F(\rho_x,\rho_y)$$

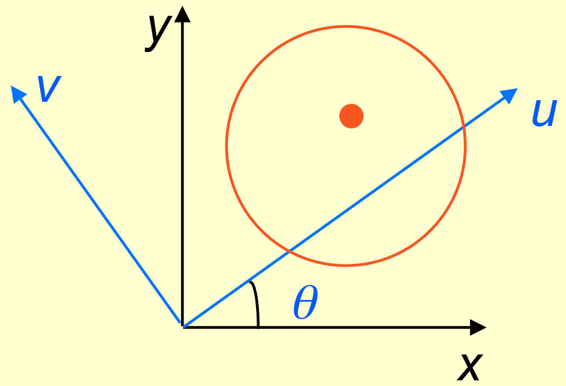
1D FT of p with respect to u = 2D FT of f in a given direction

# Analytical approach: filtered backprojection (FBP)

$$P(\rho,\theta) = F(\rho_x,\rho_y)$$

$$f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\rho_x,\rho_y)\, e^{i2\pi(x\rho_x+y\rho_y)}\, d\rho_x.d\rho_y$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(\rho,\theta)\, e^{i2\pi(x\rho_x+y\rho_y)}\, d\rho_x.d\rho_y$$




$u = x\cos\theta + y\sin\theta$

$\rho_x = \rho\cos\theta$

$\rho_y = \rho\sin\theta$

$\rho = (\rho_x^2 + \rho_y^2)^{1/2}$

$\rho\, d\rho_x d\rho_y = \rho\, d\rho\, d\theta$

$$= \int_0^{\pi}\int_{-\infty}^{+\infty} P(\rho,\theta)\, |\rho| e^{i2\pi\rho u}\, d\rho.\, d\theta$$

$$= \int_0^{\pi} p'(u,\theta)\, d\theta \quad \text{with} \quad p'(u,\theta) = \int_{-\infty}^{+\infty} P(\rho,\theta)\, \underset{\uparrow \text{Ramp filter}}{|\rho|} e^{i2\pi\rho u}\, d\rho$$

# Filtered backprojection: algorithm

$$f(x,y) = \int_0^{\pi} p'(u,\theta)\, d\theta \qquad \text{with} \qquad p'(u,\theta) = \int_{-\infty}^{+\infty} P(\rho,\theta)\, |\rho| e^{i2\pi\rho u}\, d\rho$$

sinogram

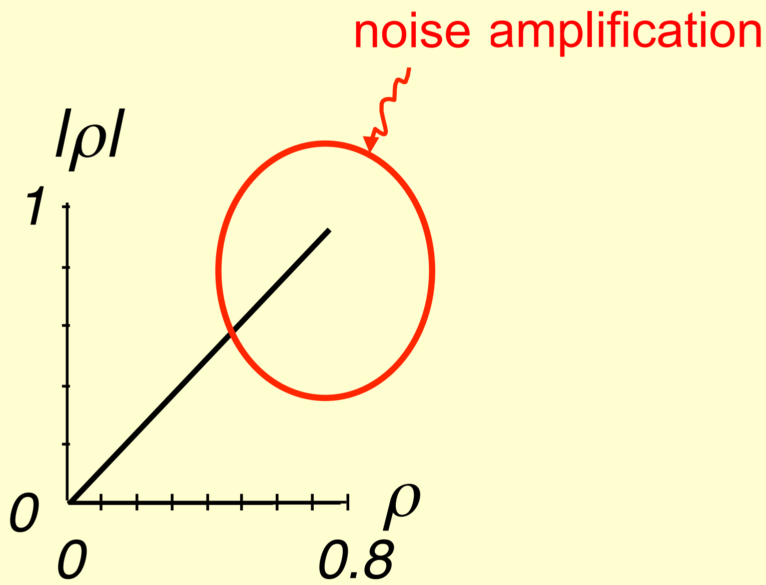
reconstructed image

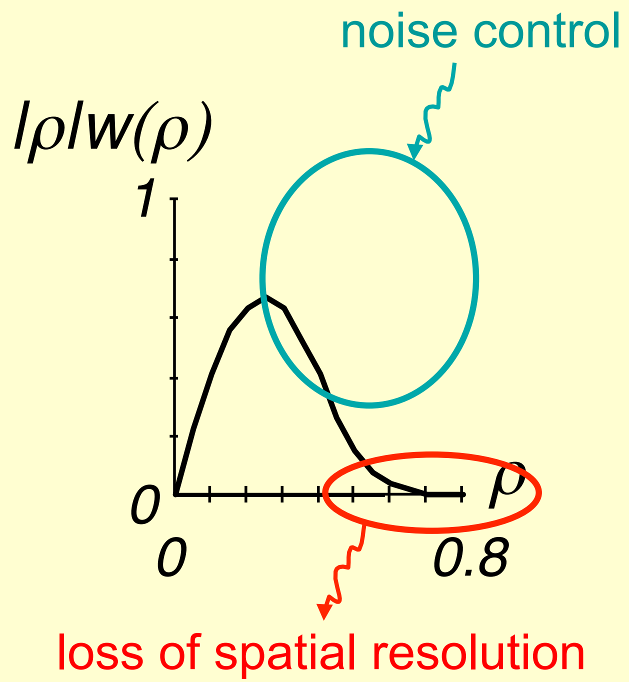
$$\begin{tikzcd}[column sep=huge, row sep=huge]
p(u,\theta) \arrow[d, "\text{1D-FT}"'] & & f(x,y) \\
P(\rho,\theta) \arrow[r, "\text{filtering}"] & |\rho|\, P(\rho,\theta) \arrow[r, "FT^{-1}"] & p'(u,\theta) \arrow[u, "\text{backprojection}"'] \\
& \text{Ramp filter} \arrow[u] &
\end{tikzcd}$$

# Filtered backprojection: beyond the Ramp filter

$$f(x,y) = \int_0^{\pi} p'(u,\theta)\, d\theta \quad \text{with} \quad p'(u,\theta) = \int_{-\infty}^{+\infty} P(\rho,\theta)\, |\rho| e^{i2\pi\rho u}\, d\rho$$

$$|\rho| \longrightarrow |\rho| w(\rho)$$



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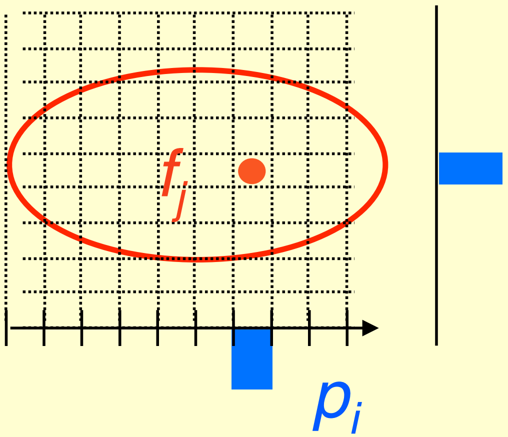
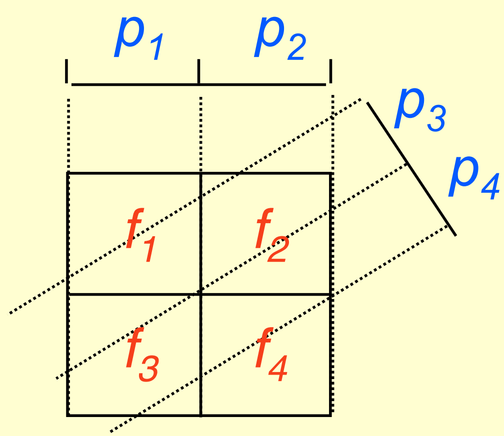
# Discrete approach: model




$$
\begin{aligned}
p_1 &= r_{11}f_1 + r_{12}f_2 + r_{13}f_3 + r_{14}f_4\\
p_2 &= r_{21}f_1 + r_{22}f_2 + r_{23}f_3 + r_{24}f_4\\
p_3 &= r_{31}f_1 + r_{32}f_2 + r_{33}f_3 + r_{34}f_4\\
p_4 &= r_{41}f_1 + r_{42}f_2 + r_{43}f_3 + r_{44}f_4
\end{aligned}
$$

$$
\begin{bmatrix} p_1\\ p_2\\ p_3\\ p_4 \end{bmatrix} = \begin{bmatrix} r_{11} & \cdots & r_{14}\\ \vdots & \ddots & \vdots\\ r_{41} & \cdots & r_{44} \end{bmatrix} \begin{bmatrix} f_1\\ f_2\\ f_3\\ f_4 \end{bmatrix}
$$

$$p = R\,f$$

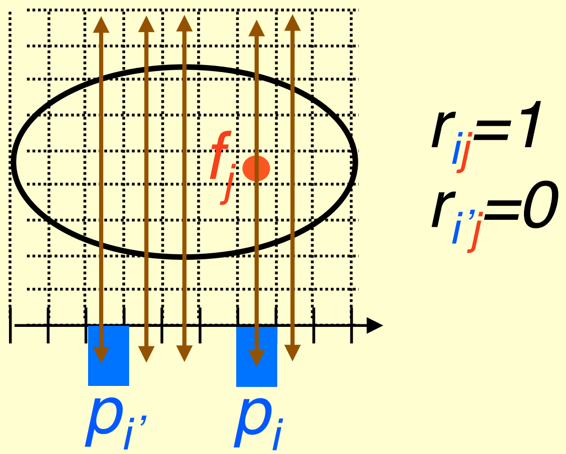
system matrix: probability that an event emitted in j be detected in LOR i

Given p and R, estimate f

# Discrete approach: calculation of R

$$p = R\,f \qquad R \text{ models the direct problem}$$

- Geometric modelling
  - intersection between each pixel and each LOR




- Physics modelling
  - spatial resolution of the detector
  - particle interactions (scatter, photoelectric absorption)

# Two classes of discrete methods

❶ Algebraic methods

❷ Statistical approaches

$$p_i = \sum_j r_{ij} f_j$$

- Generalized inverse methods

- Bayesian estimates
- Optimization of functionals

- Account for noise properties

# Iterative algorithm used in discrete methods

$$p = R f$$

$f^{0}$

$p = R f$

$\hat{p}$ $\xrightarrow{\text{comparison}}$ $\hat{p}$ versus $p$ $\xrightarrow{\text{correction}}$ $f^{1}$

define the iterative method:

additive if $f^{n+1} = f^{n} + c^{n}$

multiplicative *if* $f^{n+1} = f^{n} . C^{n}$

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# Algebraic methods

$$p = R\,f$$

Minimisation of $||p - R\,f||^2$

Several minimisation algorithms are possible to estimate a solution:

e.g., SIRT (Simultaneous Iterative Reconstruction Technique)
Conjugate Gradient
ART (Algrebraic Reconstruction Technique)

e.g., additive ART:

$$f_j^{n+1} = f_j^{n} + (p_i - p_i^{n})\, r_{ij} / \sum_k r_{ik}^2$$

$$p = R\,f$$

Probabilistic formulation (Bayes' equation):

$$proba(f|p) = proba(p|f)\ proba(f)\ /\ proba(p)$$

probability of obtaining f when p is measured | likelihood of p | prior on f | prior on p

Find a solution f maximizing proba(p|f) given a probabilistic model for p

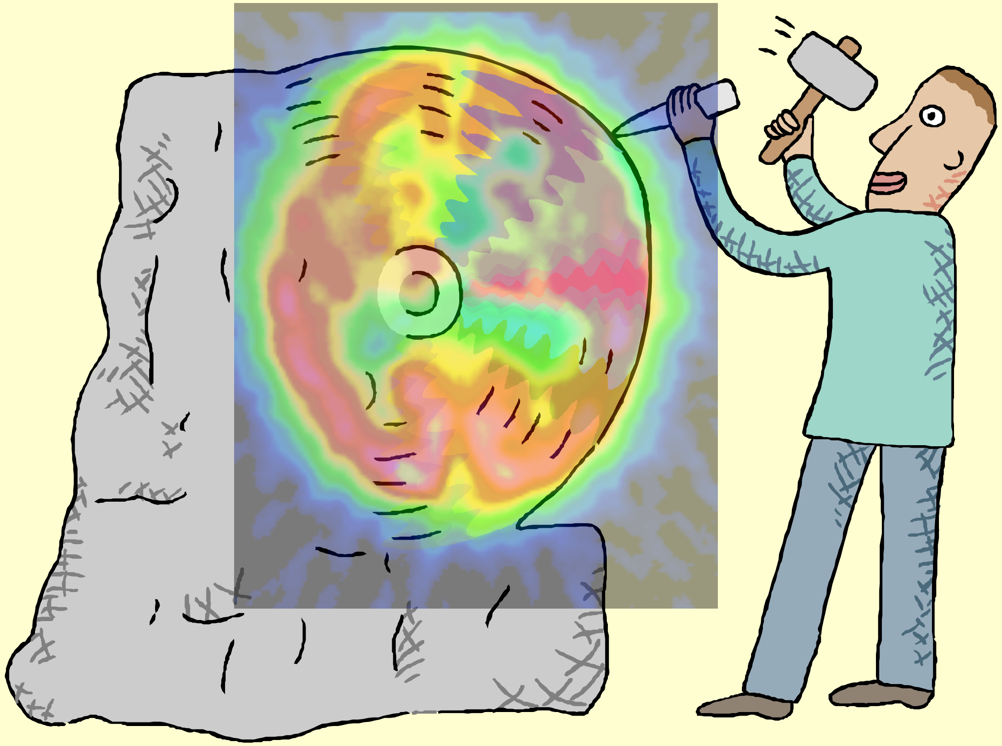
e.g., if p follows a Poisson law: $proba(p|f) = \prod_k exp(-\bar{p}_k).\bar{p}_k^{\,p_k}/p_k!$

MLEM (Maximum Likelihood Expectation Maximisation):

$$f^{n+1} = f^n \cdot R^t(p/p^n)$$

*and* OSEM (accelerated version of MLEM)

# Regularization



# Regularization

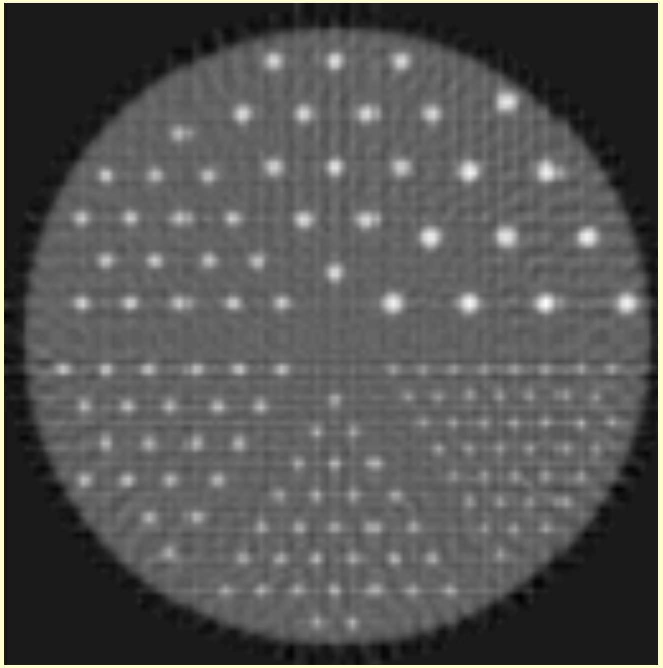
Set constraints on the solution f based on a prior

Solution f:
trade-off between
the agreement with the observed data
and
the agreement with a prior

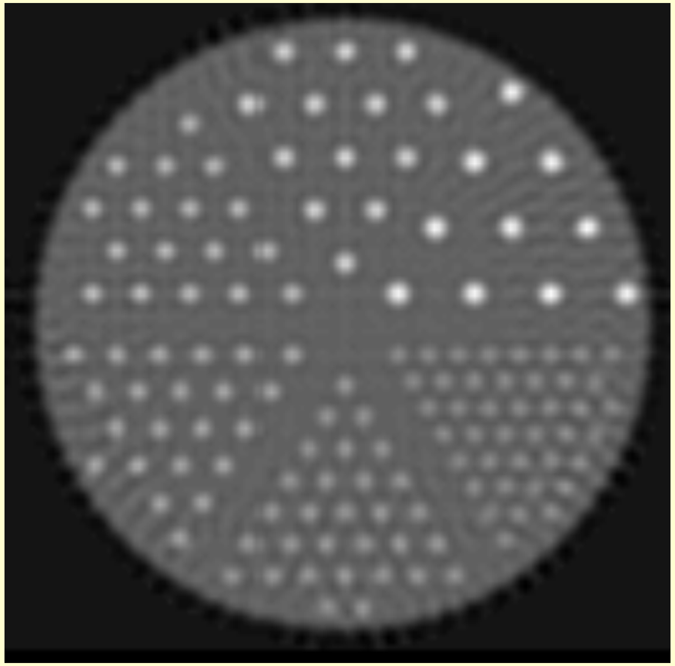
# Regularization for analytical methods

## Filtering

$$f(x,y) = \int_0^{\pi}\int_{-\infty}^{+\infty} P(\rho,\theta)\, w(\rho)|\rho|e^{i2\pi\rho u}\, d\rho$$



Ramp filter



Butterworth filter

# Regularization for discrete methods

Minimisation of $||p - R\, f||^2 + \lambda\, K(f)$

$\lambda$ controls the trade-off between
agreement with the projections and agreement with the prior

$$proba(f|p) = proba(p|f)\ proba(f) \,/\, proba(p)$$

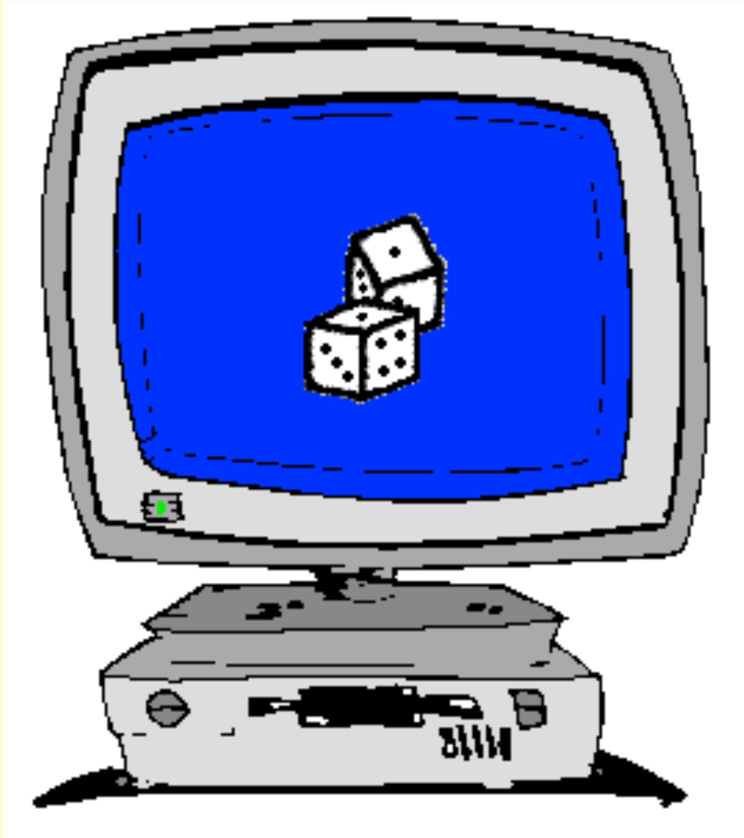
prior on f, i.e. proba(f) non uniform

Examples of priors:

- f smooth
- f having discontinuities

Conjugate Gradient gives MAP-Conjugate Gradient (Maximum A Posteriori)
MLEM gives MAP-EM

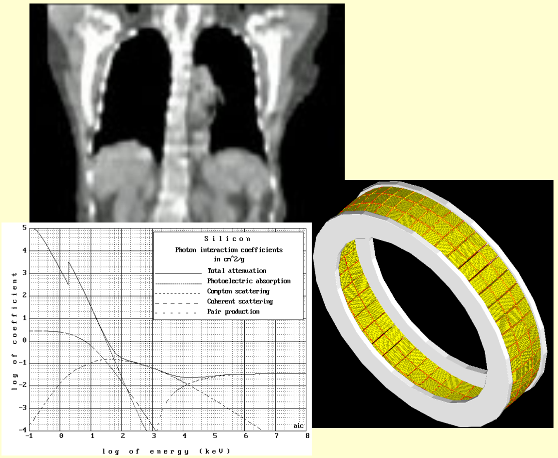
# Simulations

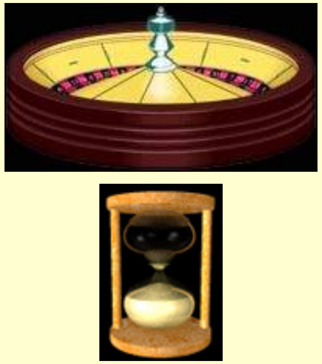


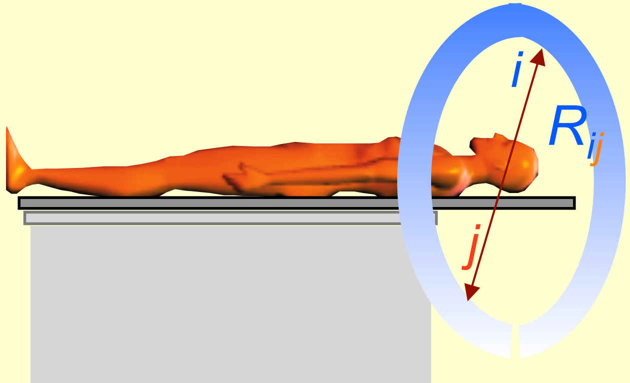
# How can simulations be used in reconstruction ?

$$p = R\, f$$

modelling R using numerical (Monte Carlo) simulations
of the imaging procedure








tissue density
and composition
+
cross-section tables of
radiation interaction
+
model of the detector

stochastic modelling
of physical interactions

probability that an
“event” in j be detected
in LOR i

↓

$R$

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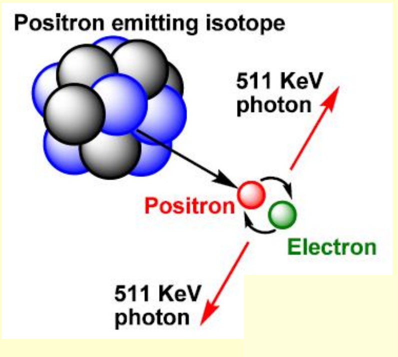
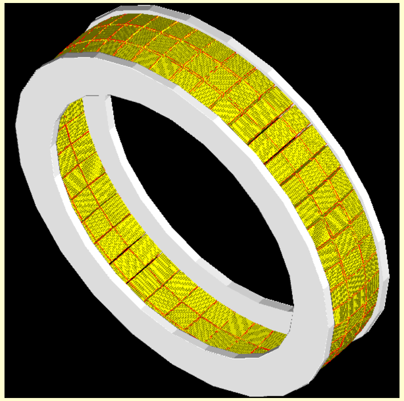
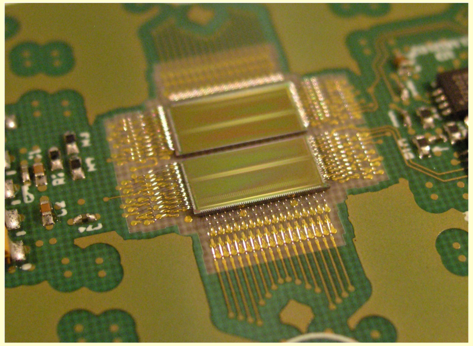
# Advantages of using Monte Carlo simulations

- Precise modelling of most phenomena involved in PET:
  - emission of positron followed by annihilation
  - stochastic interactions between particles and patient tissues
  - stochastic interactions within the detector materials
  - electronic response of the detector



- Fully 3D

**If R is accurate, then the reconstructed images will be more accurate than with an approximate R**

# Principle of MC simulations in PET

random number generator and sampling of probability density functions

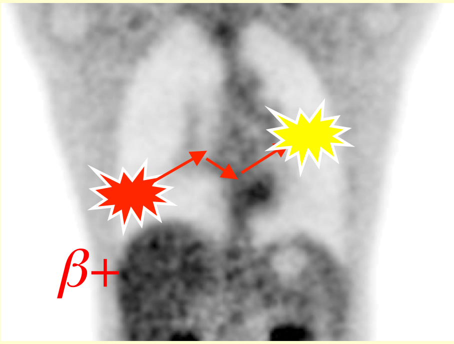
$\beta^+$ source:
geometry, decay (Emax),
activity (kBq/mL), period,
decay time, annihilation

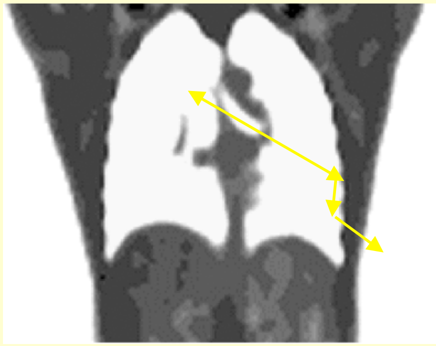
→

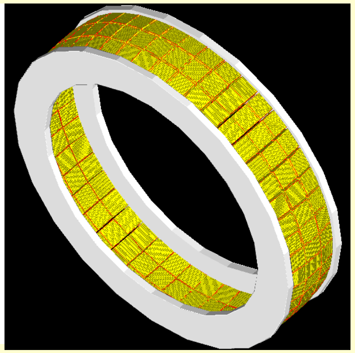
photons:
1. acolinearity, direction
2. interactions in medium
3. interactions in the detector

→

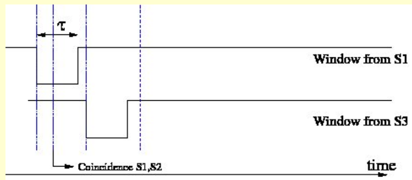
single photons:
1. energy deposit
2. light sharing
3. position calculation
4. energy calculation



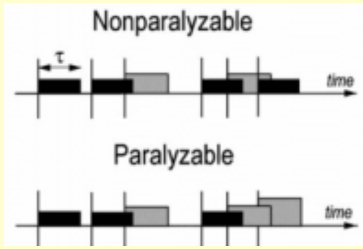





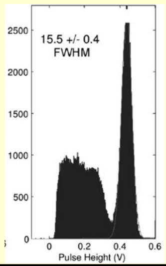
coincidences:
1. coincidence time window
2. delayed window




dead time




energy window




storage listmode data

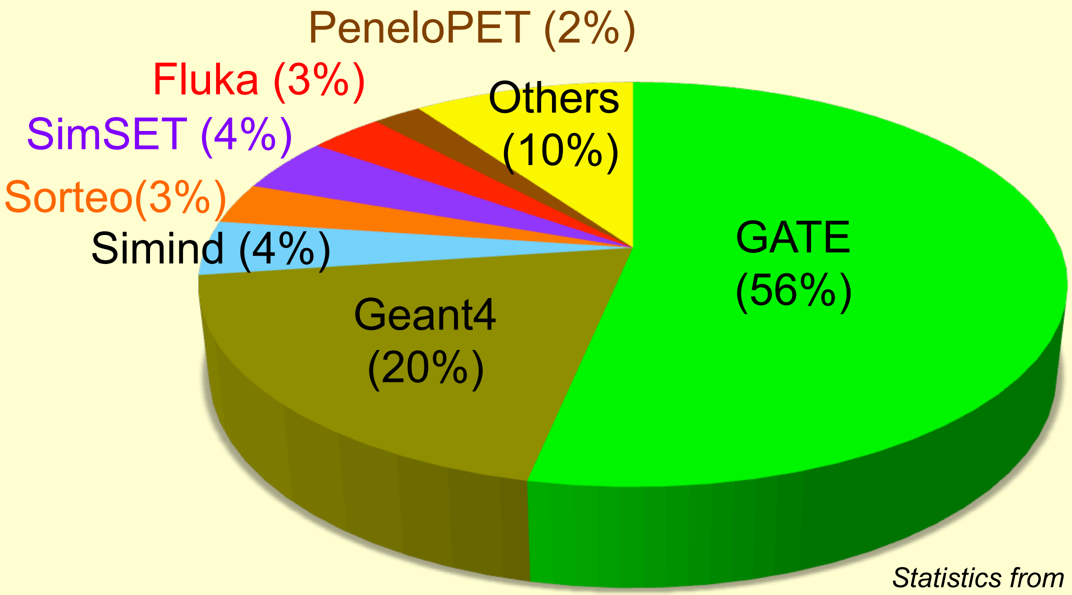
$LOR\ i_1, E_1, E'_1, t_1, LOR\ i_2, E_2, E'_2, t_2, \ldots$

IMIV Imagerie Moléculaire In Vivo

# From a practical point of view

- Monte Carlo codes modelling particle-matter interactions can be used (Geant4, EGS4, MCNPx, FLUKA, etc)
- Or, codes dedicated to MC simulations of Emission Tomography (easier to use for modelling PET acquisitions):

GATE: http://www.opengatecollaboration.org
SimSET: http://depts.washington.edu/simset/html/simset_main.html
PeneloPET: http://nuclear.fis.ucm.es/penelopet/
Sorteo: http://sorteo.cermep.fr




*Statistics from IEEE MIC 2013*

# Limitations of using MC simulations to estimate R

- In fully 3D, the R matrix is huge, typically >$10^{13}$ entries

  - factorize the matrix into several components
  - use compression techniques for storage
  - take advantage of symmetries in the scanner
  - set to 0 entries with a very low probability

- To get a sound estimate of each R entry, many events have to be simulated

  - use MC simulations to parameterize analytical functions that fit the imaging system response
  - set to 0 entries for which the statistical robustness is not ensured
  - design fast dedicated MC codes using simplifying assumptions

- Such a detailed R is not so easy to invert: not sparse, not well-conditioned, lengthy convergence of the iterative algorithm

  - use a hybrid approach that uses the most accurate R only at some iterations

# Is it worth it?

**Great Idea ☑**

**Got The Tools ☑**

**Is It Worth It ?**

- It depends on the detector design and radionuclides of interest

- Yes for “dirty” radionuclides with complicated decay schemes, ie Iodine 124, Yttrium 90

- Yes to correct for positron range for isotopes with a high positron ranges (high Emax), such as Rubidium 82 in cardiac imaging

- Yes when detector response is hard to model analytically

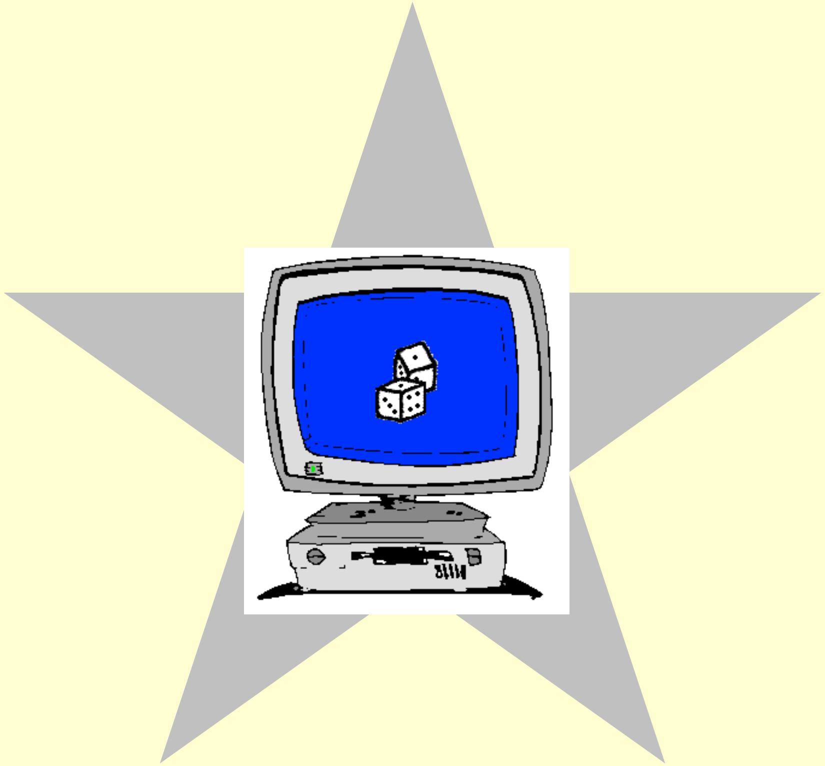
- Yes to get an accurate estimate of scatter in highly heterogeneous media

- No otherwise, analytical models, sometimes tuned based on MC simulations, work rather well in most applications

# What else MC simulations can be used for in PET?

Guidance for detector design: simulating before building

Parameterization of analytical models for corrections (scatter, non stationary PSF)



Protocol optimization: determining the impact of various detector/acquisition parameters

Evaluating the accuracy of quantification in PET: in silico experiments provide the ground truth

Calculating R for image reconstruction

# Conclusions

- Designing a new PET detector is only part of the work

- The detector has to be associated with an efficient reconstruction procedure

- Today, iterative reconstructions are preferred:
  - more flexible to model non-standard detector geometry,
  - elegantly incorporate corrections for attenuation, detector response, scatter (possibly positron range and patient motion)

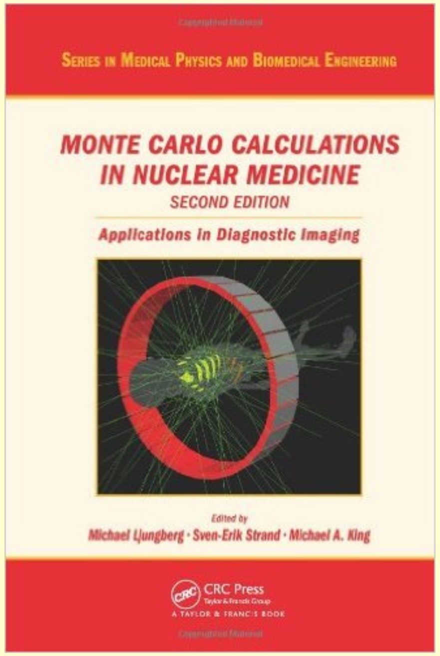
- Monte Carlo simulations can assist PET image reconstruction as it can guide the design of the R matrix system

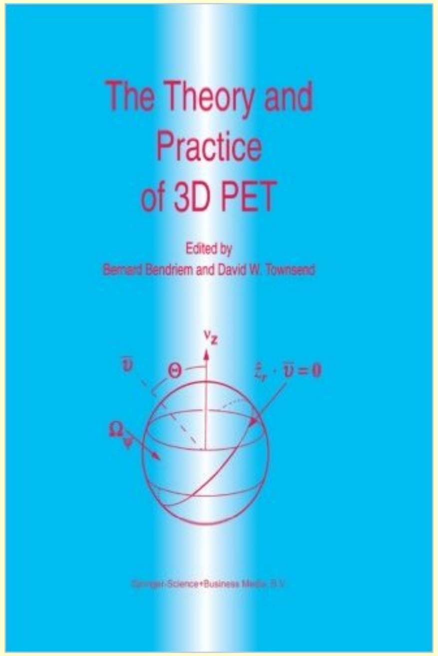
- Monte Carlo simulations are part of the toolbox of PET scientists, as they contribute to all steps of PET research (detector design, reconstruction, corrections, protocol optimization, assessment of quantification)

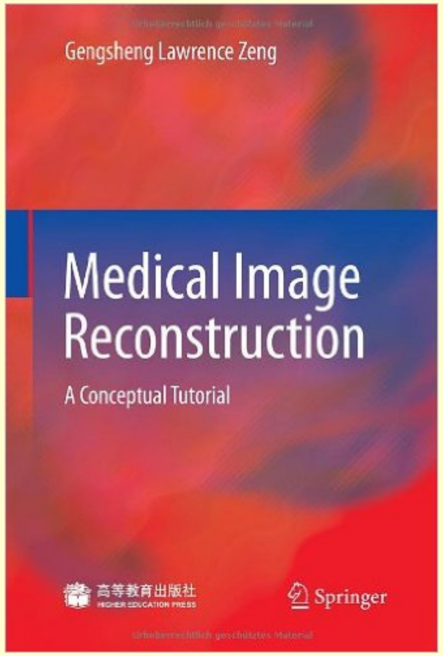
# Additional resources

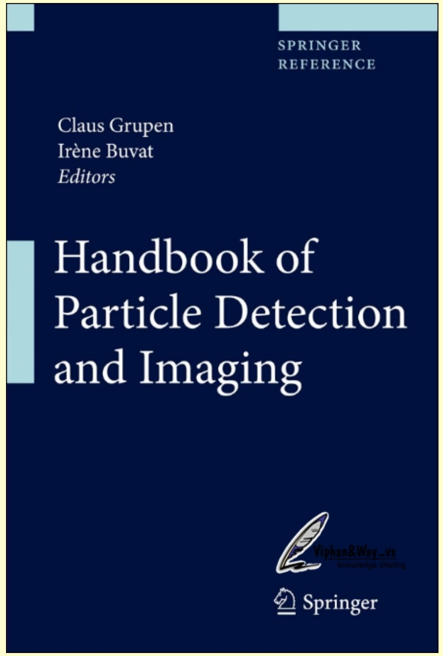
A more detailed course regarding simulations (3 h !):
http://www.guillemet.org/irene/coursem/ENTERVISION_Simulations.pdf

Several courses regarding tomographic reconstruction in PET and SPECT:
http://www.guillemet.org/irene/cours



# Announcements

GATE training course 6-8 October 2015 in Orsay, France

Check http://www.opengatecollaboration.org for registration
Training tab
Registration will close this Wednesday (September 9th)

# Principles of PET: reconstruction / simulations

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